THE TRANSMISSION OF SONIC BOOM SIGNALS INTO ROOMS THROUGH OPEN WINDOWS

PART 2: THE TIME DOMAIN SOLUTIONS

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SUMMARY

In this report, the time domain extensions of results previously reported are sought.

Expressions for pressure fields inside rooms due to a delta impulse type excitation have been obtained, both by using a normal mode type approach and by using a Helmholtz resonator analogy.

These results have been applied to the specific case of 'N'-wave type excitations.

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INTRODUCTION

Expressions were obtained in Part 1 of this series of reports (1) for the sound field inside a room due to a harmonic wave entering the room through a window.

When the incoming wave is in the form of a transient signal, the same expressions could be used to compute the spectrum of the sound field inside the room, if the spectrum of the incoming transient signal were known.

However, there is a considerable interest in obtaining a solution in the time domain. The analytical methods used to achieve this are based on either the Fourier transform or the Laplace transform techniques. A direct method of solution, using these techniques, leads to cumbersome expressions which are difficult to compute. An adaptation of the technique, similar to Mintzer's (2), however, leads to a solution which essentially consists of the multiple reflection approach (3), in which reflection at the end wall is described by means of a unit function with suitable retarded - time arguments. The series, so obtained, is seen to posess, at any given time, only a finite number of non-zero terms.

It is shown in the second chapter how this description is particularly suited to represent the earlier part of the time history of the response. This part of the response is most significant from a subjective point of view.

When the sound field in the room has become fairly well diffused, the room can be justifiably represented by a lumped acoustic element (or several elements), particularly when the attention is shifted to the lower frequencies. Therefore the Helmholtz resonator approach can be used to describe the later part of the time history. This part of the response is of interest, mainly from the structural point of view (and hence the shift of emphasis to the lower frequencies).

In Chapter 1 of this report, the mathematical theory of integral transforms, which is relevant to the present work, is reviewed. The results of the previous report are also restated in a form which is suitable for time domain computation.

In Chapter 2, response of a model room to a delta-function input is computed. It should be noted that this technique is quite general and can be used to compute the response of a room to any general transient, a blast wave for example.

In Chapter 3, the results of the previous chapter are applied to a specific case of the sonic boom problem. Detailed mathematical analysis is followed by its approximate version and numerical computations.

LIST OF SYMBOLS

a	Width of the room
A	Amplitude of the incident wave
b	Height of the room
c	Speed of sound
C_p	Lumped capacitance of the room
d	Depth of the room
e	Base of the natural logarithm
$H(t)$	Heaviside unit function
i	Square root of minus one
L	Attached mass of the window
m	Modal parameter
n	Modal parameter
$\bar{n}$	An integer representing number of reflections
p	Acoustic pressure phasor
$R_{m,n}(t,z)$	Component in the $(m,n)^{th}$ mode of the field generated inside a room due to an incoming sonic-boom type signal.
$s \equiv$	$-i\omega$
$S_{m,n}(t,z)$	Component in the $(m,n)^{th}$ mode of the field generated inside a room due to a delta function excitation
$S_{m,n}^{\bar{n},1}$, $S_{m,n}^{\bar{n},2}$, $R_{m,n}^{\bar{n},1}$, $R_{m,n}^{\bar{n},2}$	Components of $S_{m,n}$, $R_{m,n}$ above.
t	Time
x, y, z	Co-ordinates
$\alpha_{m,n}$	Modal frequency parameter
δ	Dirac delta function
$\eta_{m,n}$	Modal rise time parameter
μ	Damping parameter
ρ	Density
ρ_o	Mean density

ψ	Normalised eigenfunction
ω	Radian frequency
Ω	Natural radian frequency of resonator.

1.0 THEORY OF TRANSFORMS

1.1 The Unit and Delta Functions and their Use

The Heaviside unit function, $H(t)$, is defined by the following equation :

$$H(t) = \begin{cases} 0 , & (t < 0) \\ 1 , & (t > 0) \end{cases} \qquad (1.1.1)$$

The function is also called the unit step function.

Another function, $\delta(\tau)$, termed the Dirac delta function, may be defined by the following equation :

$$\int_0^t \delta(\tau)\, d\tau = H(t) \qquad (1.1.2)$$

It has a zero value everywhere except at $\tau = 0$.

It is interesting to note that if

$$G_H(\eta,t) = 1 - e^{-\frac{t}{\eta}} , \qquad (1.1.3)$$

and

$$G_\delta(\eta,t) = \frac{1}{\eta}\, e^{-\frac{t}{\eta}} ; \qquad (1.1.4)$$

$G_H(\eta,t)$ behaves like $H(t)$ when $\eta \to 0$ and $G_\delta(\eta,t)$ behaves like $\delta(t)$ when $\eta \to 0$. These two forms could be considered as sort of 'relaxed' forms of the unit step and delta functions.

The following properties of the delta and the unit function follow from their definition :

$$\int_{-\infty}^{\infty} f(\tau)\, \delta(t - \tau)\, d\tau = f(t) , \qquad (1.1.5)$$

and

$$\int_{-\infty}^{\infty} H(t - \tau)\, \frac{df(\tau)}{d\tau}\, d\tau = f(t). \qquad (1.1.6)$$

Physically, these equations can be interpreted as follows: a smooth continuous function of time can be considered as being built up of an infinite sequence of impulse functions, the one at time τ having amplitude $f(\tau)$, and so on. Or the function can be built up stepwise, the increase at time τ being proportional to $\frac{df}{d\tau}$.

These equations can be used when the response of a system, say $X_\delta(\tau)$, to an output, $\delta(\tau)$, is known; or if the response $X_H(\tau)$ to an input $H(\tau)$, is known. Further, it might be required to find out the response of the system for an arbitrary input $f(t)$ from these known responses.

It is seen from (1.1.5) and (1.1.6) that the required response, $X_f(t)$ is given by

$$X_f(t) = \int_{-\infty}^{\infty} f(\tau)\, X_\delta(t - \tau)\, d\tau \quad , \tag{1.1.7}$$

and

$$X_f(t) = \int_{-\infty}^{\infty} \frac{df(\tau)}{d\tau}\, X_H(t - \tau)\, d\tau \quad , \tag{1.1.8}$$

(see reference 4, p.49). An additional term is required in (1.1.8) if the function $f(t)$ is discontinuous (4, p.49).

Thus the task of finding the response of a system in the time domain can be split into two steps: step (i) find out the response of the system to either a delta impulse function or to a unit step function (ii) use the appropriate convolution integral (1.1.7) or (1.1.8). The general procedure for step (i) is described in the next section.

1.2 Fourier and Laplace Transform Technique

Assume that the response of a system to an input $e^{-i\omega t}$ is known and the response to either a delta excitation, $\delta(t)$, or a unit step excitation, $H(t)$, is required.

In these contexts, the following relationship is found to be useful :

If

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i\omega t}\, d\omega \quad , \tag{1.2.1}$$

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{+i\omega t}\, dt \; . \tag{1.2.2}$$

f is said to be the Fourier transform of F and the relationship states that, conversely, $2\pi F$ is also the Fourier transform of f.

Now, if it is known that the response of a system to an input $e^{-i\omega t}$ is $X(\omega)\, e^{-i\omega t}$, it follows that the response of the system to an input $A(\omega)\, e^{-i\omega t}$ shall be $A(\omega)\, X(\omega)\, e^{-i\omega t}$.

Further

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(t)\, e^{i\omega t}\, dt = \frac{1}{2\pi} \tag{1.2.3}$$

and therefore

$$\delta(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-i\omega t} dt . \qquad (1.2.4)$$

Therefore the response, $X_\delta(t)$, to an excitation $\delta(t)$ can be expressed as :

$$X_\delta(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X(\omega) e^{-i\omega t} d\omega . \qquad (1.2.5)$$

Similarly,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} H(t) e^{-i\omega t} dt = \frac{-1}{2\pi i \omega} , \qquad (1.2.6)$$

and therefore

$$H(t) = \int_{-\infty}^{\infty} \frac{-1}{2\pi i\omega} e^{-i\omega t} d\omega . \qquad (1.2.7)$$

Therefore the response, $X_H(t)$, to a unit impulse excitation, $H(t)$, can be expressed as :

$$X_H(t) = \int_{-\infty}^{\infty} \frac{-1}{2\pi i\omega} X(\omega) e^{-i\omega t} d\omega . \qquad (1.2.8)$$

Integrals (1.2.5) and (1.2.8) can be evaluated by the usual infinite integral techniques. It is often preferable to convert these integrals into the Laplace transform form, which can either be regarded as a generalized form of the Fourier transform or a special form of Mellin's transform.

Substitute in (1.2.5)

$$s = -i\omega , \qquad (1.2.9)$$

and

$$\bar{X}(s) = X(is) , \qquad (1.2.10)$$

to obtain

$$X_\delta(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \bar{X}(s) e^{st} dt , \qquad (1.2.11)$$

$$= L^{-1}\,\bar{X}(s) \quad , \tag{1.2.12}$$

where L^{-1} denotes 'inverse Laplace transform of'. The Laplace transform of a function $f(t)$ is given by (see reference 5, p.227) :

$$g(s) \equiv Lf(t) = \int_0^\infty e^{-st}\,f(t)\,dt \;; \tag{1.2.13}$$

and the inverse Laplace transform of $g(s)$ is given by the relation

$$f(t) \equiv L^{-1}\,g(s) = \frac{1}{2\pi i}\int_{c'-i\omega}^{c'+i\omega} g(s)\,e^{st}\,dt \;. \tag{1.2.14}$$

c' is supposed to have a vanishingly small positive value. Physically, its presence indicates the inevitable presence of dissipation in the system.

A relationship, similar to (1.2.12), holds good for $X_H(t)$.

$$X_H(t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\bar{X}(s)}{s}\;e^{st}\,dt \tag{1.2.15}$$

$$= L^{-1}\left(\frac{\bar{X}(s)}{s}\right) \quad . \tag{1.2.16}$$

Thus the problem of deducing the response of the system to either a delta or unit step impulse reduces to that of finding the inverse Laplace transform of either $\bar{X}(s)$ or $\frac{\bar{X}(s)}{s}$, where $\bar{X}(s)$ is obtained from the amplitude of the steady-state response, $X(\omega)$, by means of the substitutions (1.2.9) and (1.2.10). Bateman (5) has edited an exhaustive collection of formulae to obtain

$$L^{-1}\,\bar{X}(s) \quad \text{from } \bar{X}(s).$$

The procedure to obtain the time domain response of a system to an arbitrary input, $f(t)$, can now be summarised as follows :

FIND THE STEADY STATE RESPONSE, $X(\omega)\,e^{-i\omega t}$ TO AN INPUT OF UNIT AMPLITUDE, $e^{-i\omega t}$

↓

FORMULATE $\bar{X}(s)$ from $X(\omega)$

↓

FIND EITHER $L^{-1}\bar{X}(s)$ or $L^{-1}(\bar{X}(s)/s)$

↓

USE APPROPRIATE CONVOLUTION RELATIONSHIP TO FIND THE RESPONSE TO $f(t)$

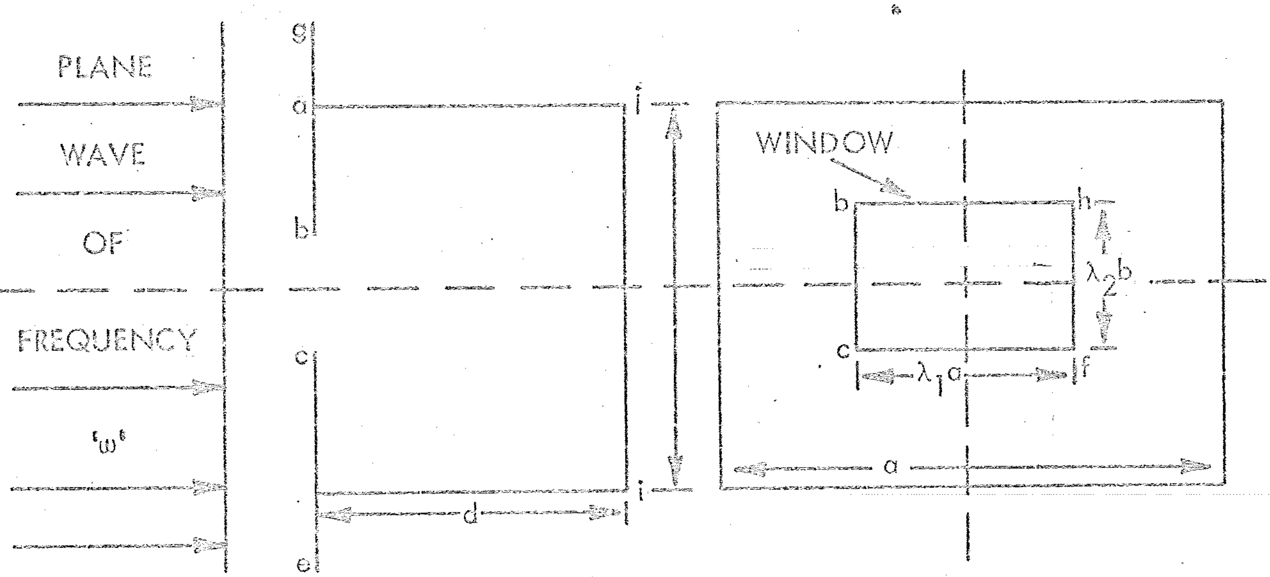


FIGURE 1 A SECTIONAL END VIEW OF THE ROOM AND THE FRONT VIEW IS SHOWN. THE PLANE HARMONIC WAVE IS NORMALLY INCIDENT ONTO THE FACE GE.

1.3 Reduction of $F_{m,n}(\omega,z)$ to the Form $\overline{F_{m,n}}(s,z)$

It has been shown (1) that the pressure field generated inside a room, sketched in Figure 1, due to an incoming wave,

$$p_i = A\, e^{-i\omega t} \quad ,$$

is given by the equation

$$P = 2A\, e^{-i\omega t} \sum_{m,n} F_{m,n}(\omega,z) \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} , \qquad (1.3.1)$$

where

$$F_{m,n}(\omega,z) = \frac{\sigma_{m,n} \cos\left[\sqrt{\frac{\omega^2}{c^2} - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}\right](d-z)}{\cos\left[\sqrt{\frac{\omega^2}{c^2} - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}\right]d - Q_{m,n} \sin\left[\sqrt{\frac{\omega^2}{c^2} - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}\right]d}$$

$$(1.3.2)$$

$\sigma_{m,n}$ is a normalising parameter which depends upon the geometry of the problem and it is independent of ω. Further

$$Q_{m,n} = \left[\sqrt{\frac{\omega^2}{c^2} - (\frac{m\pi}{a})^2 - (\frac{n\pi}{b})^2}\right] \left(\frac{ab\, \sigma^2_{m,n}}{(2-\delta_{o,n})(2-\delta_{o,n})}\right) \left(\frac{1}{M} - \frac{i}{2\pi c}\right) \times \frac{1}{(\lambda_1\lambda_2)^2}$$

$$(1.3.3)$$

The pressure field inside the room due to an incoming wave, which is represented by a delta-impulse, i.e. $p_i = \delta(t)$, is given by the inverse Laplace transform of

$$\overline{F_{m,n}}(s,z)$$

multiplied by

$$\cos\frac{m\pi x}{a} \quad \cos\frac{n\pi y}{b} \quad .$$

$\overline{F_{m,n}}(s,z)$ is obtained by substituting

$$s = -i\omega \qquad (1.3.4)$$

in the expression for $F_{m,n}(\omega,z)$.

Therefore

$$\overline{F_{m,n}}(s,z) = \frac{\sigma_{m,n} \cosh \left(\frac{d-z}{c}\right) \sqrt{s^2 + \alpha^2_{m,n}}}{\cosh \frac{d}{c} \sqrt{s^2 + \alpha^2_{m,n}} + \eta_{m,n} \left[1 + \frac{Ms}{2\pi c}\right] \sqrt{s^2 + \alpha^2_{m,n}} \sinh \frac{d}{c} \sqrt{s^2 + \alpha^2_{m,n}}} , \tag{1.3.5}$$

where

$$\alpha_{m,n} = c \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} , \tag{1.3.6}$$

and

$$\eta_{m,n} = \frac{ab \ \sigma^2_{m,n}}{(2 - \delta_{o.m})(2 - \delta_{o.n}) \ Mc[\lambda_1\lambda_2^*]^2} \tag{1.3.7}$$

Expansion of the hyperbolic functions in the equation (1.3.5) and the use of the binomial theorem reduces the equation to

$$\overline{F_{m,n}}(s,z) = \frac{\sigma_{m,n}}{1+\eta_{m,n}\left[1 + \frac{Ms}{2\pi c}\right]r} \left[\sum_{\bar{n}=0} \mu^{\bar{n}} \left(e^{-\frac{r}{c}(2\bar{n}d + z)} + e^{-\frac{r}{c}([2\bar{n} + 2]d-z)}\right)\right] \tag{1.3.8}$$

where

$$\mu \ (=\mu_{m,n}) = \frac{\eta_{m,n}\left[1 + \frac{Ms}{2\pi c}\right] r-1}{\eta_{m,n}\left[1 + \frac{Ms}{2\pi c}\right] r+1} , \tag{1.3.9}$$

$$r \ (=r_{m,n}) = \sqrt{s^2 + \alpha^2_{m,n}} . \tag{1.3.10}$$

Evaluation of the transform of $\overline{F_{m,n}}$ is discussed in the next section.

2.0 RESPONSE OF THE ROOM TO A DELTA FUNCTION IMPULSE

2.1 Transform for (0,0) Mode

When both m and n are zero, the relations (1.3.8) to (1.3.10) assume the following forms :

$$\overline{F_{(o,o)}}(s,z) = \frac{\sigma_{o,o}}{1 + \eta_{oo}(1 + \frac{Ms}{2\pi c})} \left[\sum_{\bar{n}=0} \mu^{\bar{n}}_{o,o} (e^{-\frac{s}{c}(2\bar{n}d + z)} + e^{-\frac{s}{c}[(2\bar{n}+2)d - z]} \right] , \tag{2.1.1}$$

$$\mu_{oo} = \frac{\eta_{o,o}(1 + \frac{Ms}{2\pi c})\ s - 1}{\eta_{o,o}(1 + \frac{Ms}{2\pi c})\ s + 1} , \tag{2.1.2}$$

$$r_{o,o} = s , \tag{2.1.3}$$

$$\eta_{o,o} = \frac{ab}{Mc} . \tag{2.1.4}$$

It will now be assumed that the term $\frac{Ms}{2\pi c}$, for the most significant range of s, is much smaller than 1. $\frac{2\pi c}{s}$ can be viewed as the wavelength of the spectral component of the signal under consideration, and M, the mobility, is very nearly equal to the diameter of the circular window which is equal in area to the actual window. The greater part of a representative sonic boom spectrum (and to a lesser degree the subjectively weighted spectrum) in most cases, falls in the range of validity of the assumption. Expressions are obtained by using this assumption and in section 2.2 the modifications which would be necessary, in the absence of its validity, have been discussed.

With this assumption

$$\mu_{o,o} = \frac{\eta_{o,o}\ s - 1}{\eta_{o,o}\ s + 1} . \tag{2.1.5}$$

The following relationship (6) is now to be used :

$$L^{-1}\left[\frac{\phi(s)}{(s-a')^n}\right] = e^{at} \sum_{p=1}^{n} \frac{t^{n-p}}{(p-1)!(n-p)!} \left\{ \frac{d^{p-1}}{ds^{p-1}} \phi(s) \right\}_{s=a'} + h(t) , \tag{2.1.6}$$

where h(t) refers to terms due to poles other than at $s = a'$.

Therefore

$$L^{-1}\left[\frac{(S-\frac{1}{\eta})^{n-1}}{(s+\frac{1}{\eta})^{n}}\right]$$

$$= e^{-\frac{t}{\eta}} \sum_{p=1}^{n} \frac{t^{n-p}}{(p-1)!(n-p)!} \left[1 \times (n-1)(n-2) \ldots (n-p+1)\right] \left(\frac{-2}{\eta}\right)^{n-p}$$

$$= e^{-\frac{t}{\eta}} \sum_{p=1}^{n} \frac{t^{n-p}(n-1)!}{(p-1)!\left[(n-p)!\right]^2} \left(\frac{-2}{\eta}\right)^{n-p} \quad ; \qquad (2.1.7)$$

and therefore

$$L^{-1}\left[\frac{s-(\frac{1}{\eta})^{\bar{n}}}{(s+\frac{1}{\eta})^{\bar{n}+1}}\right]$$

$$= e^{-\frac{t}{\eta}} \sum_{p=1}^{\bar{n}+1} \frac{t^{\bar{n}-p+1}\,(\bar{n}!)}{(p-1)!\left[\bar{n}+1-p)!\right]^2} \left(\frac{-2}{\eta}\right)^{\bar{n}+1-p}$$

$$= e^{-\frac{t}{\eta}} \sum_{p=1}^{\bar{n}+1} (-1)^{\bar{n}+1-p}\ \bar{n}\,C_{p-1}\ \{2^{\bar{n}+1-p}\} \frac{(t/\eta)^{\bar{n}+1-p}}{(\bar{n}+1-p)!}$$

$$= e^{-\frac{t}{\eta}} \sum_{p=1}^{\bar{n}+1} B\bar{n},p-1 \frac{(t/\eta)^{\bar{n}+1-p}}{(\bar{n}+1-p)!} \quad , \qquad (2.1.8)$$

where

$$B\bar{n},p-1 = (-1)^{\bar{n}+1-p}\ \bar{n}\,C_{p-1}\ \{2\}^{\bar{n}+1-p} \qquad (2.1.9)$$

and the binomial coefficient, $\bar{n}Cp$, is given by

$$\bar{n}Cp = \frac{\bar{n}!}{p!(\bar{n}-p)!} \quad . \qquad (2.1.10)$$

It follows that

$$L^{-1}\left[\frac{(\eta s - 1)^{\bar{n}}}{(\eta s + 1)^{\bar{n}+1}}\right]$$

$$= \frac{e^{-\frac{t}{\eta}}}{\eta} \sum_{p=1}^{\bar{n}+1} B\bar{n},_{p-1} \frac{(t/\eta)^{\bar{n}+1-p}}{(\bar{n}+1-p)!} \quad ; \tag{2.1.11}$$

or rearranging,

$$L^{-1}\left[\frac{(\eta s - 1)^{\bar{n}}}{(\eta s + 1)^{\bar{n}+1}}\right]$$

$$= \frac{e^{-\frac{t}{\eta}}}{\eta} \sum_{p=o}^{\bar{n}} B\bar{n},_{p} \frac{(t/\eta)^{\bar{n}-p}}{(n-p)!} \quad , \tag{2.1.12}$$

$$\equiv \psi_{\bar{n}}(t) \quad , \tag{2.1.13}$$

say.

Evidently,

$$B\bar{n},_p = (-1)^{\bar{n}-p}\ \bar{n}Cp\ \{2^{\bar{n}-p}\} \quad . \tag{2.1.14}$$

The translation property is now used to obtain the transform of $\overline{F_{oo}}(s,z)$.

This property can be stated as follows (5): If $L^{-1}(g(s) = f(t)$ and b' be a constant,

$$L^{-1}\left[e^{-b's}\ g(s)\right] = \begin{cases} 0 & t < b' \\ f(t-b') & t > b' \end{cases} \quad . \tag{2.1.15}$$

The definition of the unit function can be used to rewrite equation (2.1.15) as

$$L^{-1}\left[e^{-b's}\ g(s)\right] = H(t-b')\ f(t-b') \quad . \tag{2.1.16}$$

Therefore, from equation (2.1.1), (2.1.5), (2.1.13) and (2.1.16) it is seen that

$$L^{-1}\,\overline{F_{o,o}}(s,z) = \sigma_{o,o}\sum_{\bar{n}=o}^{\infty}\psi_{\bar{n}}\left(t - \frac{2\bar{n}d+z}{c}\right) H\left(t - \frac{2\bar{n}d+z}{c}\right)$$

$$+\ \psi_{\bar{n}}\left(t - \frac{(2\bar{n}+2)d-z}{c}\right) H\left(t - \frac{(2\bar{n}+2)d-z}{c}\right) . \qquad (2.1.17)$$

The form of the equation (1.3.1) and the discussion in section 1.2 make it apparent that the expression (2.1.17) when multiplied by two will represent the contribution of the (o,o) mode to the field generated inside the room due to an incoming 'delta' impulse. Let us denote this contribution by the symbol $S_{o,o}$ so that

$$S_{o,o}(t,z) \equiv 2L^{-1}\,\overline{F(s,z)} \quad . \qquad (2.1.18)$$

This mode corresponds to the axial wave which moves forwards and backwards between the end and front walls. The progress of the wave can be readily traced.

The term $\psi_o(t - \frac{z}{c})$ comes into operation at any point z when a time of $\frac{z}{c}$ has elapsed. The term is, in fact, a distorted form of $\delta(t - \frac{z}{c})$ - distorted due to the passage of the incident wave through the aperture. The next term, $\psi_o(t - \frac{2d-z}{c})$, comes into operation after the wave has reflected from the end wall, and so on.

It is evident, due to the presence of the Heaviside functions, that at any given instant only a finite number of terms have a non-zero value.

2:2 Effect of the Term $(\frac{Ms}{2\pi c})$

It is recalled that

$$\eta_{o,o} = \frac{ab}{Mc} \quad , \qquad (2.2.1)$$

and the window area

$$ab\lambda_1\lambda_2 \simeq \frac{\pi}{4} M^2 \quad , \qquad (2.2.2)$$

Therefore

$$\eta_{o,o} = \frac{\pi M}{4\lambda_1\lambda_2 c} \quad . \qquad (2.2.3)$$

Further, if

$$M' \equiv \frac{M}{2\pi c} \quad , \tag{2.2.4}$$

$$\eta_{o,o} = \frac{\pi^2 M'}{2\lambda_1\lambda_2} \quad . \tag{2.2.5}$$

In the extreme case $\lambda_1\lambda_2 \to 1$ and therefore $\eta_{o,o}$ is about five times greater in magnitude than M'. In general, $\eta_{o,o}$ would be many more times greater. This observation can be used in the following argument:

$$L^{-1} \left(\frac{[\eta_{o,o}M's^2 + \eta_{o,o}s-1]^{\bar{n}}}{[\eta_{o,o}M's^2 + \eta_{o,o}s+1]^{\bar{n}+1}}\right)$$

$$\simeq L^{-1} \quad \frac{(\eta_{o,o}s-1)^{\bar{n}}(M's + 1)^{\bar{n}}}{(\eta_{o,o}S+1)^{\bar{n}+1}(M's+1)^{\bar{n}+1}}$$

$$= L^{-1} \quad \frac{1}{Ms'+1} \quad \frac{(\eta_{o,o}s-1)^{\bar{n}}}{(\eta_{o,o}s+1)^{\bar{n}+1}} \quad . \tag{2.2.6}$$

$L^{-1} \dfrac{(\eta_{o,o}s-1)^{\bar{n}}}{(\eta_{o,o}s+1)^{\bar{n}+1}}$ was found in section 2.1, equation (2.1.12). A revised form of $\psi_{\bar{n}}(t)$ can be obtained by substituting the expression (2.2.6) for its value. This can be done by using the convolution integral for the transform of a product (6).

Such a procedure leads to the following expression :

$$\psi_{\bar{n}}(t) = \sum_{p=o}^{\bar{n}} B_{\bar{n},p} \left(\frac{M'}{M'-\eta_{o,o}}\right)^{\bar{n}-p} \left\{\frac{e^{-t/\eta} - e^{t/M'}}{\eta_{o,o} - M'}\right\} \left(\frac{(t/\eta')^{\bar{n}-p}}{(\bar{n}-p)!} + \ldots \frac{(t/\eta')}{1} + 1\right) \tag{2.2.7}$$

where

$$M' = \frac{2\pi M}{c} \quad , \tag{2.2.8}$$

$$\eta' = \frac{\eta_{o,o}M'}{M'-\eta_{o,o}} \quad . \tag{2.2.9}$$

Therefore $$\frac{e^{-t/\eta_{o,o}}}{\eta_{o,o}}$$ is modified to the form

$$\frac{e^{-t/\eta_{o,o}} - e^{-t/M'}}{\eta_{o,o} - M'} .$$

The modification is slight and is confined to a small region near the origin. Other terms undergo a similar modification which is relatively small, when M' is much smaller than $\eta_{o,o}$.

2.3 Formulation of an Integral to Obtain Inverse Laplace Transforms for Higher Order Modes

It was shown in the previous sections that

$$L^{-1}\left[\left(\frac{(\eta_{o,o}s-1)^{\bar{n}}}{(\eta_{o,o}s+1)^{\bar{n}+1}}\right)\left[e^{-\frac{s}{c}(2\bar{n}d+z)} + e^{-\frac{s}{c}([2\bar{n}+2]d-z)}\right]\right]$$

$$= \frac{e^{-\left[\frac{t-(2\bar{n}d+z)/c}{\eta_{o,o}}\right]}}{\eta_{o,o}} \sum_{p=o}^{\bar{n}} B_{\bar{n},p} \frac{\left(\frac{t-(2\bar{n}d+z)/c}{\eta_{o,o}}\right)^{\bar{n}-p}}{(\bar{n}-p)!} H\left(t - \frac{(2\bar{n}d+z)}{c}\right)$$

$$+ \frac{e^{-\left[\frac{t-([2\bar{n}+2]d-z)/c}{\eta_{o,o}}\right]}}{\eta_{o,o}} \sum_{p=o}^{\bar{n}} B_{\bar{n},p} \frac{\left(\frac{t-([2\bar{n}+2]d-z)/c}{\eta_{o,o}}\right)^{\bar{n}-p}}{(\bar{n}-p)!} H\left(t - \frac{(2n+2)d-z}{c}\right)$$

(2.3.1)

To evaluate the inverse Laplace transform of $\bar{F}_{m,n}(s,z)$, the inverse transform of the same function, as in the L.H.S. above, is required with the substitution of $r_{m,n}$ see equation (1.3.9) for s and the substitution of $\eta_{m,n}$ for $\eta_{o,o}$. (Note that the same assumption as in the previous chapter, that $\frac{Ms}{2\pi c} << 1$, is once again made).

A substitution of $\eta_{m,n}$ in the equation (2.3.1) would leave its form unaltered. For the modification due to $r_{m,n}$, a theorem regarding the inverse transforms is required (see 2, p.227). It can be stated as follows: If

$$L^{-1}\, g(s) = f(t) \tag{2.3.2}$$

$$L^{-1} g(r_{m,n}) = f(t) - \alpha_{m,n} \int_0^t f\left[(t^2 - u^2)^{\frac{1}{2}}\right] \times J_1(\alpha_{m,m} u)\, du \,, \tag{2.3.3}$$

where

$$r_{m,n} = \sqrt{s^2 + \alpha_{m,n}^{\;2}} \,, \tag{2.3.4}$$

and $\alpha_{m,n}$ is a constant. In the context of the present work $\alpha_{m,n}$ is given by the equation (1.4.6). J_1 is the Bessel function of the first order.

For $\bar{n} = 0$, the first term of (2.3.1) corresponds to the following relationship :

$$L^{-1} \frac{e^{-\frac{s}{c}z}}{1 + \eta_{o,o} S} = \frac{e^{-\frac{(t - \frac{z}{c})}{\eta_{o,o}}}}{\eta_{o,o}} H(t - \frac{z}{c}) \,. \tag{2.3.5}$$

It can be deduced therefore that

$$L^{-1} \frac{e^{-\frac{s}{c}z}}{1 + \eta_{m,n} S} = \frac{e^{-\frac{(t - \frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}} H(t - \frac{z}{c}) \,. \tag{2.3.6}$$

The relationship (2.3.3) can be used to show that

$$L^{-1} \frac{e^{-\frac{r_{m,n}}{c} z}}{1 + \eta_{m,n}\, r_{m,n}} = \frac{e^{-\frac{(t - \frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}} (H(t - \frac{z}{c})$$

$$- \alpha_{m,n} \int_0^t \frac{e^{-\frac{1}{\eta_{m,n}} (\sqrt{t^2 - u^2} - \frac{z}{c})}}{\eta_{m,n}} J_1(\alpha_{m,n} u) \quad H(\sqrt{t^2 - u^2} - \frac{z}{c})\, du \,. \tag{2.3.7}$$

Denote this term by

$$\bar{S}_{m,n}^{o,1} (t,z).$$

The definition of the unit step function can be used to rewrite the integral involved in the term.

Therefore

$$\bar{S}^{o,1}_{m,n}(t,z) = \left[\frac{e^{-\frac{1}{\eta_{m,n}}(t - \frac{z}{c})}}{\eta_{m,n}} - \alpha_{m,n} \int_0^{\sqrt{t^2 - \frac{z^2}{c^2}}} e^{-\frac{1}{\eta_{m,n}}(\sqrt{t^2 - u^2} - \frac{z}{c})} J_1(\alpha_{m,n} u)\, du \right] H(t - \frac{z}{c}) \qquad (2.3.8)$$

Substitute

$$\theta \equiv \frac{\sqrt{t^2 - u^2}}{\eta_{m,n}} , \qquad (2.3.9)$$

so that $u = \sqrt{t^2 - \eta_{m,n}^2 \theta^2}$ and $du = \eta_{m,n}^2 \frac{\theta d\theta}{\sqrt{t^2 - \eta_{m,n}^2 \theta}}$; and as u goes from 0 to $\sqrt{t^2 - \frac{z^2}{c^2}}$, θ goes from $t/\eta_{m,n}$ to $\frac{z}{c\eta_{m,n}}$. Therefore

$$\bar{S}^{o,1}_{m,n}(t,z) = \frac{e^{-\frac{1}{\eta_{m,n}}(t - \frac{z}{c})}}{\eta_{m,n}} - \alpha_{m,n}\, \eta_{m,n}\, e^{\frac{z}{c\eta_{m,n}}} H(t - \frac{z}{c}) \int_{\frac{z}{c\eta_{m,n}}}^{t/\eta_{m,n}} \theta e^{-\theta} \frac{J_1(\alpha_{m,n}\sqrt{t^2 - \eta_{m,n}^2 \theta^2})}{\sqrt{t^2 - \eta_{m,n}^2 \theta^2}} d\theta$$

(2.3.10)

The evaluation of this integral will be attempted by using the method of successive integration by parts. The choice of which part to integrate and which part to differentiate is made by judging the results for convergence. Roughly speaking, when $\alpha_{m,n}\ \eta_{m,n}$ is small, the Bessel function does not oscillate very rapidly and therefore it can be arranged to go with the part which is to be differentiated and vice versa. An expression will be obtained later in Section 2.4 which is appropriate when $\alpha_{m,n}\eta_{m,n} < 1$ and one in Section 2.5 which is appropriate when $\alpha_{m,n}\eta_{m,n} > 1$.

2.4 Evaluation of the Integral when $\alpha_{m,n}\eta_{m,n} < 1$

Let us denote the integral in the expression (2.3.10) by the symbol D, so that

$$D \equiv \int_{\frac{z}{c\eta_{m,n}}}^{t/\eta_{m,n}} \theta e^{-\theta} \frac{J_1(\alpha_{m,n}\sqrt{t^2 - \eta_{m,n}^2\theta^2})}{\sqrt{t^2 - \eta_{m,n}^2\,\theta^2}}\, d\theta \quad . \tag{2.4.1}$$

Now it is readily verified that

$$\frac{\partial}{\partial\theta}\; \frac{J_1(\alpha_{m,n}\sqrt{t^2 - \eta_{m,n}^2\theta^2})}{\sqrt{t^2 - \eta_{m,n}^2\theta^2}} = \alpha_{m,n}\,\eta^2\; \frac{\theta J_2(\alpha_{m,n}\sqrt{t^2 - \eta_{m,n}^2\theta^2})}{(t^2 - \eta_{m,n}^2\,\theta^2)}\,, \tag{2.4.2}$$

and

$$\int \theta e^{-\theta} d\theta \;=\; -e^{-\theta}\left[\theta + 1\right] . \tag{2.4.3}$$

Therefore

$$D = \left[-e^{-\theta}\left[\theta + 1\right] \;\frac{J_1(\alpha_{m,n}\sqrt{t^2 - \eta_{m,n}^2\theta^2})}{\sqrt{t^2 - \eta_{m,n}^2\,\theta^2}} \right]_{\frac{z}{c\eta_{m,n}}}^{\frac{t}{\eta_{m,n}}}$$

$$+\; \alpha_{m,n}\,\eta_{m,n}^2 \int_{\frac{z}{c\,\eta_{m,n}}}^{\frac{t}{\eta_{m,n}}} e^{-\theta}\;\left[\theta + 1\right]\theta\; \frac{J_2(\alpha\sqrt{t^2 - \eta_{m,n}^2\theta^2})}{(t^2 - \eta_{m,n}^2\;\theta^2)}\, d\theta \tag{2.4.4}$$

Further, since

$$\frac{\partial}{\partial\theta}\ \frac{J_2(\alpha_{m,n}\sqrt{t^2 - \eta^2_{m,n}\theta^2})}{(t^2 - \eta^2_{m,n}\theta^2)} = \alpha\eta^2_{m,n}\ \frac{\theta J_3(\alpha\sqrt{t^2 - \eta^2_{m,n}\theta^2})}{[\sqrt{t^2 - \eta^2_{m,n}\ \theta^2}]^3}\ , \qquad (2.4.5)$$

and

$$\int e^{-\theta}\ [\theta^2 + \theta]\ d\theta = -e^{-\theta}[\theta^2 + 3\theta + 3]\ , \text{ etc;}$$

(2.4.4) can be rewritten as

$$-D = \Bigg[e^{-\theta}[\theta^2 + 1]\ \frac{J_1(\alpha_{m,n}\ \sqrt{t^2 - \eta^2_{m,n}\ \theta^2})}{\sqrt{t^2 - \eta^2_{m,n}\ \theta^2}}$$

$$+ \alpha\eta^2_{m,n}\ e^{-\theta}\ [\theta^2 + 3\theta + 3]\ \frac{J_2(\alpha_{m,n}\ \sqrt{t^2 - \eta^2_{m,n}\ \theta^2})}{(t^2 - \eta^2_{m,n}\ \theta^2)}$$

$$+ \alpha^2\eta^4_{m,n}\ e^{-\theta}[\theta^3 + 6\theta^2 + 9\theta + 9]\ \frac{J_3(\alpha_{m,n}\ \sqrt{t^2 - \eta^2_{m,n}\ \theta^2})}{(\sqrt{t^2 - \eta^2_{m,n}\ \theta^3})^3}$$

$$+ \alpha^3\eta^6_{m,n}\ \ e^{-\theta}[\theta^4 + 10\theta^3 + 59\theta^2 + 87\theta + 87]\ \frac{J_4\alpha_{m,n}\sqrt{t^2 - \eta^2_{m,n}\ \ \theta^2}}{(t^2 - \eta^2_{m,n}\ \theta^2)} + \ldots \Bigg]^{\frac{t}{\eta_{m,n}}}_{\frac{z}{c\eta_{m,n}}}$$

(2.4.6)

Now since $J_m(z) \to \dfrac{z^m}{m!2^m}$ as $z \to 0$, the following limits will hold good:

$$\lim_{\theta\to t/\eta_{m,n}}\ \frac{J_1(\alpha_{m,n}\ \sqrt{t^2 - \eta^2_{m,n}\theta^2})}{\sqrt{t^2 - \eta^2_{m,n}\ \theta^2}} = \frac{\alpha_{m,n}}{2}\ , \qquad (2.4.7)$$

$$\lim_{\theta \to t/\eta_{m,n}} \frac{J_2(\alpha_{m,n}\sqrt{t^2 - \eta_{m,n}^2\ \theta^2})}{(t^2 - \eta_{m,n}^2\ \theta^2)} = \frac{\alpha_{m,n}^2}{8} , \qquad (2.4.8)$$

$$\lim_{\theta \to t/\eta_{m,n}} \frac{J_3(\alpha_{m,n}\sqrt{t^2 - \eta_{m,n}^2\ \theta^2})}{(\sqrt{t^2 - \eta_{m,n}^2\ \theta^2})^3} = \frac{\alpha_{m,n}^3}{48} , \qquad (2.4.9)$$

etc. Therefore

$$-D\alpha_{m,n}\ \eta_{m,n}\ e^{\frac{z}{c\eta_{m,n}}}\ H(t - \frac{z}{c})$$

$$= H(t - \frac{z}{c})\ \{\frac{e^{\frac{-(t - \frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}} \left[\frac{\alpha_{m,n}^2\ \eta_{m,n}^2}{2}\ (\frac{t}{\eta_{m,n}} + 1) + \frac{\alpha_{m,n}^4\ \eta_{m,n}^4}{8}\ (\frac{t^2}{\eta_{m,n}^2} + \frac{3t}{\eta_{m,n}} + 3) + ..\right]$$

$$- (\frac{\alpha_{m,n}^2}{c} + \alpha_{m,n\ m,n})\ \frac{J_1(\alpha_{m,n}\ \sqrt{t^2 - \frac{z^2}{c^2}})}{\sqrt{t^2 - \frac{z^2}{c^2}}} - \eta_{m,n}$$

$$\times \left(\frac{\alpha_{m,n}^2 z^2}{c^2} + 3\alpha_{m,n}\eta_{m,n}\ (\frac{\alpha_{m,n} z}{c}) + 3\alpha_{m,n}^2\ \eta_{m,n}^2\right) \times \frac{J_2(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}})}{(t^2 - \frac{z^2}{c^2})} + \ldots\}$$

$$(2.4.10)$$

Now if

$$B_1(t,\frac{z}{c}) = \frac{2J_1(\alpha_{m,n}\ \sqrt{t^2 - \frac{z^2}{c^2}})}{\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}}} , \qquad (2.4.11)$$

so that

$$B_1(t,\frac{z}{c}) = 1 , \quad \text{for } t = \frac{z}{c} , \qquad (2.4.12)$$

$$\bar{S}^{o,1}_{m,n}(t,z) = H\left(t - \frac{z}{c}\right)\left\{ \left[\frac{e^{-\left(\frac{t-\frac{z}{c}}{m,n}\right)}}{\eta_{m,n}} \right] - \frac{\alpha^2_{m,n}\eta^2_{m,n}}{2} \left[\frac{B_1(t,\frac{z}{c})}{\eta_{m,n}} \left(\frac{z}{c\eta_{m,n}} + 1\right) - \frac{e^{-\frac{(t-\frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}} \left(\frac{t}{\eta_{m,n}} + 1\right) \right] \right\} \tag{2.4.13}$$

Terms neglected in (2.4.11) have their coefficients outside the square brackets starting with $\alpha^4_{m,n}\eta^4_{m,n}$ or more.

It can be noted that all terms except the first one,

$$\left[\frac{e^{-\frac{(t-\frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}} \right], \quad \text{vanish at } t = z/c.$$

2.5 Evaluation of the Integral when $\alpha_{m,n}\eta_{m,n} > 1$

Consider again the integral denoted by D. Since

$$\frac{\partial}{\partial\theta} J_o\left(\alpha_{m,n}\sqrt{t^2 - {}^{2}_{m,n}\theta^2}\right) = \alpha_{m,n}\eta^2_{m,n}\theta \, \frac{J_1(\alpha\sqrt{t^2 - \eta^2_{m,n}\theta^2})}{\sqrt{t^2 - \eta^2_{m,n}\theta}}, \tag{2.5.1}$$

$$\int \theta \frac{J_1(\alpha_{m,n}\sqrt{t^2 - \eta^2_{m,n}\theta^2})}{\sqrt{t^2 - \eta^2_{m,n}\theta^2}} \, d\theta = \frac{1}{\alpha\eta^2_{m,n}} J_o\left(\alpha_{m,n}\sqrt{t^2 - {}^{2}_{m,n}\theta^2}\right). \tag{2.5.2}$$

Therefore

$$D = \left[\frac{e^{-\theta}}{\alpha\eta^2_{m,n}} J_o\left(\alpha\sqrt{t^2 - \eta^2_{m,n}\theta^2}\right) \right]_{\frac{z}{c\eta_{m,n}}}^{\frac{t}{\eta_{m,n}}} \tag{2.5.3}$$

$$- \frac{1}{\alpha_{m,n}\eta^2_{m,n}} \int_{z/c\eta_{m,n}}^{t/\eta_{m,n}} e^{-\theta} J_o\left(\alpha_{m,n}\sqrt{t^2 - \eta^2_{m,n}\theta^2}\right) d\theta.$$

Further,

$$\frac{\partial}{\partial\theta}\,\alpha_{m,n}\sqrt{t^2-\eta_{m,n}^2\,\theta^2}\quad J_1\,(\alpha\sqrt{t^2-\eta_{m,n}^2\,\theta^2})$$

$$= -\alpha_{m,n}^2\,\eta_{m,n}^2\,\theta^2\quad J_o(\alpha_{m,n}\sqrt{t^2-\eta_{m,n}^2\,\theta^2})\quad , \qquad (2.5.4)$$

and therefore

$$-\int \theta\, J_o(\alpha\sqrt{t^2-\eta_{m,n}^2\theta^2})\, d\theta \;=\; \frac{\sqrt{t^2-\eta_{m,n}^2\theta^2}\; J_1(\alpha_{m,n}\sqrt{t^2-\eta_{m,n}^2\theta^2})}{\alpha_{m,n}\,\eta_{m,n}^2} \qquad (2.5.5)$$

Thus it is seen that

$$\bar{S}_{m,n}^{o,1}(t,z) \;=\; H(t-\frac{z}{c})\,\{-\frac{J_o(\alpha_{m,n}\sqrt{t^2-\frac{z^2}{c^2}})}{\eta_{m,n}} + \frac{\sqrt{t^2-\frac{z^2}{c^2}}\;J_1(\alpha_{m,n}\sqrt{t^2-\frac{z^2}{c^2}})}{\alpha_{m,n}\,\eta_{m,n}^3\,(\frac{z}{c\eta_{m,n}})}$$

$$-\frac{(\frac{z}{c\eta_{m,n}}+1)(t^2-\frac{z^2}{c^2})J_2(\alpha_{m,n}\sqrt{t^2-\frac{z^2}{c^2}})}{\alpha_{m,n}^2\,\eta_{m,n}^5\,(\frac{z}{c\eta_{m,n}})^2} + \ldots + \frac{e^{-\frac{(t-\frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}}\} \qquad (2.5.6)$$

2.6 The Helmholtz-Resonator Approach

Imagine a rigid piston to occupy the area represented by the window in Figure 1. Let this piston possess a mass of

$$m' = \frac{\rho_o S^2}{M} \quad ,$$

where S is the area of the window and M the mobility. This mass is approximately equal to the 'inertial' mass associated with the window. Now let

$$L = \frac{m'}{S} \qquad (2.6.1)$$

represent the mass per unit area of this piston. Further, associate a damping parameter R with this piston which should represent the dissipation per unit area of its surface. This dissipation is assumed to be equal to the corresponding total dissipation in the open window problem. The causes of this dissipation are, of course, the absorption of sound due to various causes inside the resonator, particularly at the walls and in the effective neck, and the losses due to the re-radiation out into the open. An account of the losses associated with resonators has been given by Ingard (7). It is essential to note that, particularly at low frequencies, such losses are small.

To the hypothetical piston, we further attach a capacitance,

$$C_p \; \left(= \frac{V}{\rho_o c^2} \times \frac{1}{S}\right) ,$$

which takes into account the springiness of the air inside the room. Essentially, it represents the units of the piston-displacement required to produce a uniform pressure-rise of unity in the room. Now if the pressure in the incident wave for $z < 0$ is represented by $N(t - \frac{z}{c})$, the blocked reflected pressure at $z = 0$ is given by $2N(t)$. Therefore, if $\xi(t)$ be the displacement of the piston under the influence of such a disturbance, the relevant differential equation of motion may be written as

$$L\,\ddot{\xi}(t) + R\,\dot{\xi}(t) + \frac{\xi(t)}{C_p} = 2\,N(t) \quad . \qquad (2.6.2)$$

A Laplace transform of both the sides of the equation (2.6.2) results in

$$\left[Ls^2 + Rs + \frac{1}{C_p}\right]\mathcal{L}(\xi(t)) = 2\mathcal{L}(N(t)) \quad . \dagger \qquad (2.6.3)$$

And therefore

$$\xi(t) = \mathcal{L}^{-1}\left[\frac{2\mathcal{L}(N(t))}{Ls^2 + Rs + (1/C_p)}\right] \quad . \qquad (2.6.4)$$

But

$$\mathcal{L}^{-1}\;\frac{1}{Ls^2 + Rs + (1/C_p)} = \frac{e^{-\mu t}\,\sin\Omega t}{L\Omega} \qquad (2.6.5)$$

where

$$\mu = \frac{R}{2L} \quad , \qquad (2.6.6)$$

$$\Omega^2 = \frac{1}{LC_p} - \mu^2 \quad , \qquad (2.6.7)$$

see Bateman (5, pp.229).

† The symbol $\mathcal{L}$ is used for the Laplace transform, in this section, to distinguish from the length L.

The rule for the inverse transform of a product can be used to obtain, from equations (2.6.4) and (2.6.5),

$$\xi(t) = \int_{-\infty}^{t} \frac{e^{-\mu(t-\tau)} \sin \Omega(t-\tau)}{L\Omega} \left[2\ N(\tau)\right] d\tau \ . \qquad (2.6.8)$$

Therefore, the response to a delta impulse is

$$\xi(t) = \frac{2e^{-\mu t} \sin \Omega t}{L\ \Omega} \ . \qquad (2.6.9)$$

However, the general form of expression (2.6.8) is also quite useful, as seen in the next chapter.

3.0 RESPONSE OF THE ROOM TO AN 'N' WAVE

3.1 Introduction

The expressions derived in the previous chapter are of a general nature. Once the time history of an incoming transient is known, convolution formulae can be used to obtain the response of a room to that transient.

This technique has been illustrated in this chapter by choosing a specific example of an 'N' wave type signal.

It should be noted that certain parts of the discussion which follows involve some approximations which are relevant only to idealized sonic boom type signals.

3.2 Response of the (0,0) Mode

Let us now consider the response of the axial modes to an N wave excitation given by

$$\begin{aligned} I(t) &= 0 \ , && (t < 0) \\ &= A(1 - \frac{2t}{T}) \ , && (0 \leq t < T) \\ &= 0 \ . && (t \geq T) \qquad (3.2.1) \end{aligned}$$

Symbolically, I(t) can be represented as

$$I(t) = A \left[1 - \frac{2t}{T}\right] \ H(t)\ (1 - H(t - T)). \qquad (3.2.2)$$

The response in the (0,0) mode, R_{oo}, due to I(t) is given by the convolution integral:

$$R_{o,o}(t,z) = \int_{-\infty}^{t} S_{o,o}(\tau,z)\, I(t-\tau)\, dt$$

$$= A \int_{-\infty}^{t} S_{o,o}(\tau,z) \left[1 - \frac{2(t-\tau)}{T}\right] H(t-\tau) \left[1 - H(t-\tau-T)\right] dt. \tag{3.2.3}$$

The various terms which contribute to $S_{o,o}(t,z)$ can be distinguished from one another by the use of superscripts as follows :

$$\bar{S}^{\bar{n},1}_{o,o}(t,z) \equiv 2\,\sigma_{o,o}\,\psi_{\bar{n}}\left(t - \frac{2\bar{n}d + z}{c}\right) H\left(t - \frac{2\bar{n}d + z}{c}\right), \tag{3.2.4}$$

and

$$\bar{S}^{\bar{n},2}_{o,o}(t,z) \equiv 2\,\sigma_{o,o}\,\psi_{\bar{n}}\left(t - \frac{2n + 2)d-z}{c}\right) H\left(t - \frac{(2\bar{n} + 2)d-z}{c}\right). \tag{3.2.5}$$

The corresponding components of the response $R_{o,o}(t,z)$ can also be denoted in a similar fashion :

$$\bar{R}^{\bar{n},1}_{o,o}(t,z) = \int_{-\infty}^{t} \bar{S}^{\bar{n},1}_{o,o}(\tau,z)\, I(t-\tau)\, d\tau, \tag{3.2.6}$$

and

$$\bar{R}^{\bar{n},2}_{o,o}(t,z) = \int_{-\infty}^{t} \bar{S}^{\bar{n},2}_{o,o}(\tau,z)\, I(t-\tau)\, dt. \tag{3.2.7}$$

Therefore

$$S_{o,o}(t,z) = \sum_{\bar{n}=o}^{\infty} \bar{S}^{\bar{n},1}_{o,o}(t,z) + \bar{S}^{\bar{n},2}_{o,o}(t,z) \tag{3.2.8}$$

and

$$R_{o,o}(t,z) = \sum_{\bar{n}=o}^{\infty} \bar{R}^{\bar{n},1}_{o,1}(t,z) + \bar{R}^{\bar{n},2}_{o,o}(t,z). \tag{3.2.9}$$

Further let

$$\bar{z}^{\bar{n},1} = z + 2\bar{n}d \quad , \tag{3.2.10}$$

and

$$\bar{z}^{\bar{n},2} = -z + (2\bar{n} + 2)d. \tag{3.2.11}$$

Now, for any $\bar{z}$

$$\int_{\bar{z}/c}^{t} \psi_{\bar{n}} \left(\tau - \frac{\bar{z}}{c}\right) \left[1 - \frac{2(t - \tau)}{T}\right] d\tau$$

$$= \sum_{p=o}^{\bar{n}} B\bar{n},p \int_{z'/c}^{t} \frac{e^{\frac{-(\tau - \frac{\bar{z}}{c})}{\eta_{o,o}}}}{\eta_{o,o}} \frac{\left[\left[\tau - \frac{\bar{z}}{c}\right]/\eta_{o,o}\right]^{\bar{n}-p}}{(\bar{n} - p)!} \left[1 - \frac{2(t-\tau)}{T}\right] dt. \tag{3.2.12}$$

And, therefore, if

$$\theta \equiv \frac{\tau - \frac{\bar{z}}{c}}{\eta_{o,o}} \tag{3.2.13}$$

$$\frac{\bar{R}^{\bar{n},1}_{o,o}(t,z)}{2A\ \sigma_{o,o}} = \sum_{p=o}^{\bar{n}} B\bar{n},p \left[\left(\frac{1-2(t - \frac{\bar{z}}{c})}{T}\right) \int_{o}^{\frac{t - \frac{\bar{z}}{c}}{\eta_{o,o}}} e^{-\theta} \frac{\theta^{\bar{n}-p}}{(\bar{n}-p)!} d\theta \right.$$

$$\left. + \frac{2\eta_{o,o}}{T} (\bar{n}-p+1) \int_{o}^{t - \frac{z}{c}} e^{-\theta} \frac{\theta^{\bar{n}-p+1}}{(\bar{n}-p+1)!} d\theta \right] \left(\frac{\bar{z}}{c} \leq t < T + \frac{\bar{z}}{c}\right) . \tag{3.2.14}$$

By means of successive integration, the following identity can be readily verified :

$$\int e^{-\theta} \frac{\theta^n}{n!} d\theta \equiv -e^{-\theta} \left[1 + \theta + \frac{\theta^2}{2!} + \ldots\ldots \frac{\theta^n}{n!}\right] \tag{3.2.15}$$

It is observed that the bracketed expression represents the first (n+1) terms in the expansion of e^{θ} . Thus for $\theta << 1$ the integral will assume a value of -1. For large n the value of the integral will be very nearly equal to -1 even at $\theta = 1$. Thereafter for $\theta >> \bar{n}$ the integral vanishes to zero.

Substituting (3.2.15) in (3.2.14)

$$\frac{\bar{R}^{\bar{n},1}_{o,o}(\bar{z},t)}{2A\ \sigma_{o,o}} = \sum_{p=o}^{\bar{n}} B_{\bar{n},p}\ \left(1 - 2(t - \frac{z}{c})/T\right)\ \left(1 - e^{-(t - \frac{z}{c})\eta_{o,o}}\right.$$

$$\times \left[\frac{\left[(t-\bar{z}/c)/\eta_{o,o}\right]^{\bar{n}-p}}{(\bar{n} - p)!} + \frac{\left[(t - \bar{z}/c)/\eta_{o,o}\right]^{\bar{n}-p-1}}{(\bar{n}-p-1)!} + \ldots + 1\right]\Bigg)$$

$$+ \sum_{p=o}^{\bar{n}} B_{\bar{n},p}\ (\bar{n}-p-1)\ \frac{2\eta_{o,o}}{T}\left[1 - e^{-(t - \frac{\bar{z}}{c})/\eta_{o,o}}\right]$$

$$\times \left[\frac{\left((t - \frac{z}{c})/\eta_{o,o}\right)^{\bar{n}-p+1}}{(\bar{n}-p+1)!} + \frac{\left((t - \frac{\bar{z}}{c})/\eta_{o,o}\right)^{\bar{n}-p}}{(\bar{n}-p)!} + \ldots + 1\right],$$

$$\frac{\bar{z}}{c} < t < T + \frac{\bar{z}}{c}\ . \qquad (3.2.16)$$

Now

$$\sum_{p=o}^{\bar{n}} B_{\bar{n},p} = \sum_{p=o}^{\bar{n}} (-1)^{\bar{n}-p}\ \bar{n}C_p\ \{2^{\bar{n}-p}\}$$

$$= (-1)^{\bar{n}} \sum_{p=o}^{\bar{n}} \bar{n}C_p\ 2^{\bar{n}-p}\ (-1)^p$$

$$= (-1)^{\bar{n}}\ \{2-1\}^{\bar{n}}$$

$$= (-1)^{\bar{n}}\ . \qquad (3.2.17)$$

Also
$$\sum_{p=o}^{\bar{n}} B_{\bar{n},p}\ (\bar{n}-p+1)$$

$$= \sum_{p=o}^{\bar{n}} (-1)^{\bar{n}-p}\ \left\{\frac{\bar{n}!}{(\bar{n}-p)!\ p!}\right\}\ ([\bar{n}-p] + [1])\ 2^{\bar{n}-p}$$

$$= (-1)^{\bar{n}} \sum_{p=o}^{\bar{n}} 2\bar{n} \left\{\frac{(\bar{n}-1)!}{(\bar{n}-p-1)!\,p!}\right\} 2^{\bar{n}-p-1} \times (-1)^p + \frac{n!}{(n-p)!\,p!} 2^{\bar{n}-p} \times (-1)^p$$

$$= (-1)^{\bar{n}} \left(2\bar{n}(2-1)^{\bar{n}-1} + (2-1)^{\bar{n}}\right)$$

$$= (-1)^{\bar{n}} (2\bar{n} + 1) \; . \tag{3.2.18}$$

These identities can be used to obtain a simplified version of $\bar{R}^{\bar{n},1}_{o,o}$, (for $\frac{\bar{z}}{c} < t < T + \frac{\bar{z}}{c}$) :

$$\frac{\bar{R}^{\bar{n},1}_{o,o}(t,z)}{2A\,\sigma_{o,o}} = (-1)^{\bar{n}} \left(1 - \frac{2}{T}\left(t - \frac{\bar{z}^{\bar{n},1}}{c}\right)\right) \times \left(1 - K_{\bar{n}}\left(\frac{t}{\eta_{o,o}} - \frac{\bar{z}^{\bar{n},1}}{\eta_{o,o}c}\right)\right)$$

$$+ (-1)^{\bar{n}} \frac{2\eta_{o,o}}{T} (2\bar{n}+1) \left(1 - L_{\bar{n}}\left(\frac{t}{\eta_{o,o}} - \frac{\bar{z}^{\bar{n},1}}{\eta_{o,o}c}\right)\right), \quad \left(\frac{\bar{z}^{\bar{n},1}}{c} < t < T + \frac{\bar{z}^{\bar{n},1}}{c}\right)$$

$$\tag{3.2.19}$$

where

$$K_{\bar{n}}(\theta) \equiv (-1)^{\bar{n}} \sum_{p=o}^{\bar{n}} B_{\bar{n},p}\; e^{-\theta}\left[1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots \frac{\theta^{\bar{n}-p}}{(\bar{n}-p)!}\right] ;$$

$$\tag{3.2.20}$$

and

$$L_{\bar{n}}(\theta) \equiv (-1)^{\bar{n}} \sum_{p=o}^{\bar{n}} \frac{B_{\bar{n},p}}{(2\bar{n}+1)} (\bar{n}-p+1)\; e^{-\theta}\left[1 + \theta + \frac{\theta^2}{2!} + \frac{\theta^3}{3!} + \ldots \frac{\theta^{\bar{n}-p+1}}{(\bar{n}-p+1)!}\right]$$

$$\tag{3.2.21}$$

For $\bar{n}=0$, both $K_{\bar{n}}(\theta)$ and $L_{\bar{n}}(\theta)$ become $e^{-\theta}$. Thus the expression (3.2.19) approaches a value of $(-1)^{\bar{n}} \left(1 - \frac{2}{T}\left(t - \frac{z^{\bar{n},1}}{c}\right) + \frac{2\eta}{T}\right)$, with a rise time of η. For higher values of $\bar{n}$, the terms in square brackets for the expansion of (3.2.20) and (3.2.21) approach the expansion of e^{θ}. It is seen from this and the equations (3.2.17) and (3.2.18) that, for small values of θ, $K_{\bar{n}}$ and $L_{\bar{n}}$ approach unity. Thus $\bar{R}^{\bar{n},1}_{o,o}$ remains very small until sufficient time has elapsed when $K_{\bar{n}}$ and $L_{\bar{n}}$ approach zero and the expression (3.2.19) can assume its steady state value. Figure 2 shows a plot of $K_{\bar{n}}(\theta)$ and Figure 3 shows a plot of $L_{\bar{n}}(\theta)$. Both these functions start with a value of unity and approach zero. The higher the value of $\bar{n}$, the longer it takes for the functions to do so.

The behaviour, described so far, can be viewed as an extension of the analogous switching-circuit transient-behaviour. Physically, this represents the phase distortion caused due to the presence of the window. As the wave passes through the window and later as it reflects every time from the front wall, the sound field is distorted. It will be seen in the next chapter that the different modes are distorted to a different extent, thus causing a 'dispersion' effect.

The procedure to obtain an expression for $\bar{R}^{\bar{n},2}_{o,o}$ is identical to that to obtain the expression (4.6.19) for $\bar{R}^{\bar{n},1}_{o,o}$. Therefore

$$\frac{\bar{R}^{\bar{n},2}_{o,o}(t,z)}{2A\sigma_{o,o}} = (-1)^{\bar{n}}\left(1 - \frac{2}{T}\left(t - \frac{\bar{z}^{\bar{n},2}}{c}\right)\right) \times \left\{1 - K_{\bar{n}}\left(\frac{t}{\eta_{o,o}} - \frac{\bar{z}^{\bar{n},2}}{\eta_{o,o}c}\right)\right)$$

$$+ (-1)^{\bar{n}}\frac{2\eta_{o,o}}{T}(2\bar{n}+1) \times \left[1 - L_{\bar{n}}\left(\frac{t}{\eta_{o,o}} - \frac{\bar{z}^{\bar{n},2}}{\eta_{o,o}c}\right)\right], \left(\frac{\bar{z}^{\bar{n},2}}{c} < t < T + \frac{\bar{z}^{\bar{n},2}}{d}\right)$$

(3.2.22)

When $t > \frac{\bar{z}}{c} + T$, the term $(1-H(t-\tau-T))$ dictates the lower limit of the integral in the equation (3.2.3). Consequently, the following expression is obtained for $\bar{R}^{\bar{n},1}_{o,o}(t,z)$:

$$\frac{\bar{R}^{\bar{n},1}_{o,o}(t,z)}{2A\sigma_{o,o}} = \int_{t-T}^{t} \psi_{\bar{n}}\left(\tau - \frac{\bar{z}^{\bar{n},1}}{c}\right)\left[1 - \frac{2(t-\tau)}{T}\right] dt$$

$$= \sum_{p=o}^{\bar{n}} B_{\bar{n},p} \int_{t-T}^{t} e^{\frac{-(\tau - \frac{\bar{z}^{\bar{n},1}}{c})}{\eta_{o,o}}} \frac{1}{\eta_{o,o}} \frac{\left(\left[t - \frac{\bar{z}^{\bar{n},1}}{c}\right]/\eta_{o,o}\right)^{\bar{n}-p}}{(\bar{n}-p)!}\left[1 - \frac{2(t-\tau)}{T}\right] dt$$

(3.2.23)

With the same substitution as before, i.e.

$$\theta \equiv \frac{\tau - \frac{\bar{z}}{c}}{\eta_{o,o}},$$

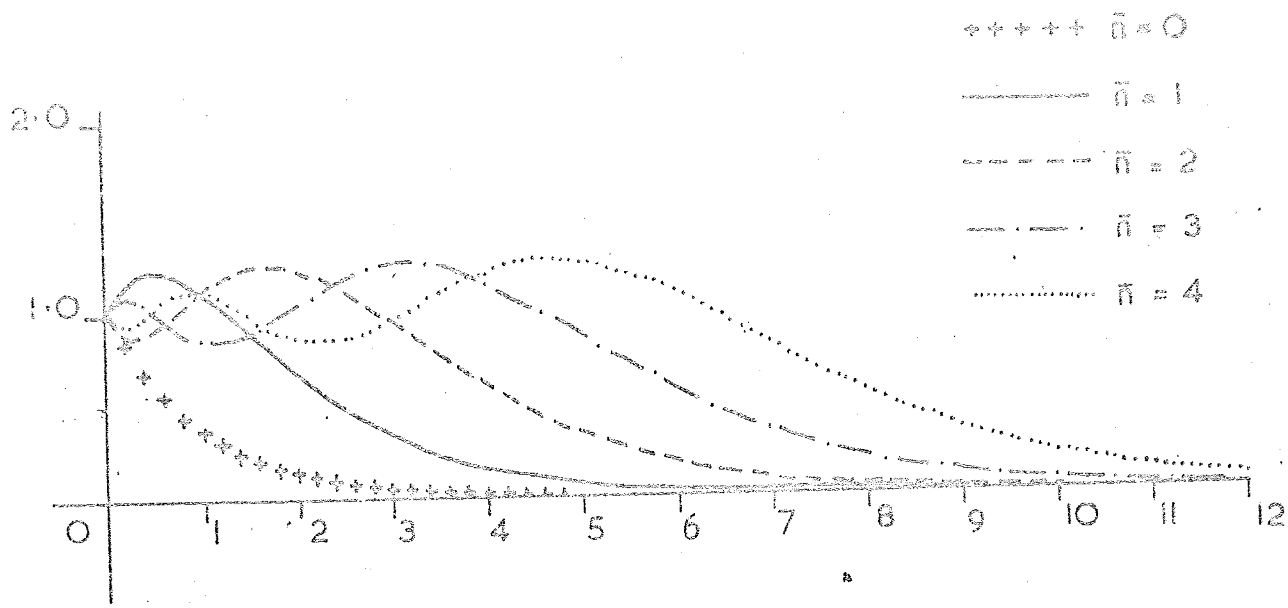


FIGURE 2 $K\bar{n}(\Theta)$

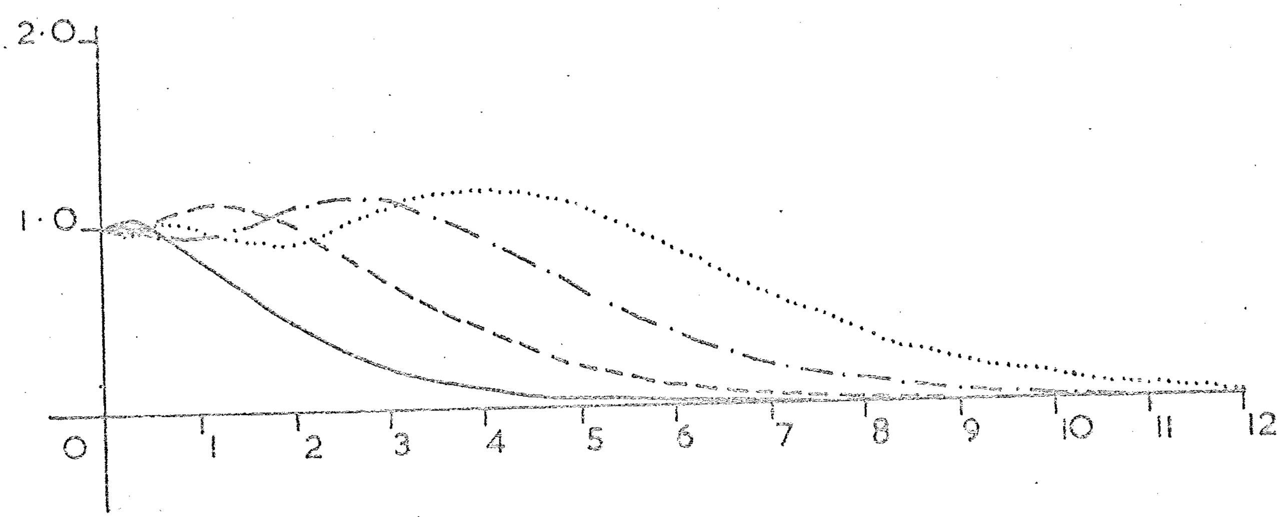


FIGURE 3 $L\bar{n}(\Theta)$

A PLOT OF THE FUNCTIONS $K\bar{n}(\theta)$ and $L\bar{n}(\theta)$ FOR VARIOUS VALUES OF $\bar{n}$: $\bar{n}$ REPRESENTS THE NUMBER OF REFLECTIONS FROM THE WALL CONTAINING THE WINDOW.

$$\frac{R^{\bar{n},1}_{(o,o)0}(z,t)}{2A\sigma_{o,o}} = \sum_{p=o}^{\bar{n}} B_{\bar{n},p} \left[\left(1 - \frac{2(t - \frac{z^{\bar{n}-1}}{c})}{T}\right) \times \right.$$

$$\left. \int_{\frac{t - T - \frac{z^{\bar{n},1}}{c}}{\eta_{o,o}}}^{\frac{t - \frac{z^{\bar{n}-1}}{c}}{\eta_{o,o}}} e^{-\theta} \frac{\theta^{\bar{n}-p}}{(\bar{n}-p)!} \, d\theta + \frac{2\eta_{o,o}}{T} (\bar{n}-p+1) \int_{\frac{t - T - \frac{z^{\bar{n},1}}{c}}{\eta_{o,o}}}^{\frac{t - \frac{z^{\bar{n},1}}{c}}{\eta_{o,o}}} e^{-\theta} \frac{\theta^{\bar{n}-p+1}}{(\bar{n}-p+1)!} \, d\theta . \right]$$

(3.2.24)

Substitution of (3.2.15) in the above expression leads to

$$\frac{\bar{R}^{\bar{n},1}_{o,o}(t,z)}{2A\sigma_{o,o}} = (-1)^{\bar{n}} \left(1 - \frac{2}{T}\left(t - \frac{z^{\bar{n},1}}{c}\right)\right) \left[-K_{\bar{n}} \left(\frac{t - \frac{z^{\bar{n},1}}{c}}{\eta_{o,o}}\right) + K_{\bar{n}} \left(\frac{t - T - z^{\bar{n},1}/c}{\eta_{o,o}}\right) \right]$$

$$+ (-1)^{\bar{n}} (2\bar{n}+1) \frac{2\eta_{o,o}}{T} \left[- L_{\bar{n}} \left(\frac{t - \frac{z^{\bar{n},1}}{c}}{\eta_{o,o}}\right) + L_{\bar{n}} \left(\frac{t - T - z^{\bar{n},1}/c}{\eta_{o,o}}\right) \right] , \quad \left(t > T + \frac{z^{\bar{n},1}}{c}\right) .$$

(3.2.25)

The square bracketed terms now represent decay after a time of T has elapsed since 'switching'.

It is advantageous to combine (3.2.25) and (3.2.19) in one form by modifying the definition of $K_{\bar{n}}$ and $L_{\bar{n}}$. This can be expressed as follows :

$$\frac{\bar{R}^{\bar{n},1}_{o,o}(t,z)}{2A\sigma_{o,o}} = (-1)^{\bar{n}} \left(1 - \frac{2}{T}\left(t - \frac{z^{\bar{n},1}}{c}\right)\right) \times \left[- K_{\bar{n}} \left(\frac{t - \frac{z^{\bar{n},1}}{c}}{\eta_{o,o}}\right) + K_{\bar{n}} \left(\frac{t-T-z^{\bar{n},1}/c}{\eta_{o,o}}\right) \right]$$

$$+ (-1)^{\bar{n}} (2\bar{n}+1) \frac{2\eta_{o,o}}{T} \left[- L_{\bar{n}} \left(\frac{t - \frac{z^{\bar{n},1}}{c}}{\eta_{o,o}}\right) - L_{\bar{n}} \left(\frac{t-T-z^{\bar{n},1}/c}{\eta_{o,o}}\right) \right]$$

(3.2.26)

where

$$K_{\bar{n}}(\theta) = 1, \quad \text{if} \quad \theta \leqslant 0$$
$$= (-1)^{\bar{n}} \sum_{p=o}^{\bar{n}} B_{\bar{n},p}\, e^{-\theta} \left[1 + \theta + \dots \frac{\theta^{\bar{n}-p}}{(\bar{n}-p)!}\right] \text{ if } \quad \theta \geqslant 0 \;;$$

(3.2.27)

and

$$L_{\bar{n}}(\theta) = 1, \quad \text{if} \quad \theta \leqslant 0$$
$$= (-1)^{\bar{n}} \sum_{p=o}^{\bar{n}} \frac{B_{\bar{n},p}(\bar{n}-p+1)}{(2\bar{n}+1)} e^{-\theta} \left[1 + \theta + \dots \frac{\theta^{\bar{n}-p+1}}{(\bar{n}-p+1)!}\right] \text{ if } \theta \geqslant 0.$$

(3.2.28)

When $t \leqslant \frac{z^{\bar{n},1}}{c}$, the square brackets amount identically to zero. For the period $\frac{z^{\bar{n},1}}{c} \leqslant t < T + \frac{z^{\bar{n},1}}{c}$, the expression (3.2.19) is obtained, and so on.

Evidently, the expression obtained by substituting the superscript $\bar{n},2$ in equation (3.2.26) instead of $\bar{n},1$, also holds good. Therefore, equation (3.2.9) can be used to obtain $R_{o,o}(t,z)$ which is the response in the (o,o) mode to an N-wave type excitation given by equation (3.2.2). The procedure to evaluate the response in higher modes will be outlined in the next chapter. It will show that under certain conditions, the predominant contribution in some of the modes follows the same pattern as for the (o,o) mode (with the evident substitution of $\eta_{m,n}$ for $\eta_{o,o}$, etc.)

The advantage of the expression (3.2.26) lies in that it is very easy to compute numerically and also it can be interpreted in simple physical terms. This was possible because of the deliberate expansion of $\bar{F}_{o,o}(s,z)$, aimed at a multiple-reflection type description of the room response (See Doak et al (3), Mintzer (2).

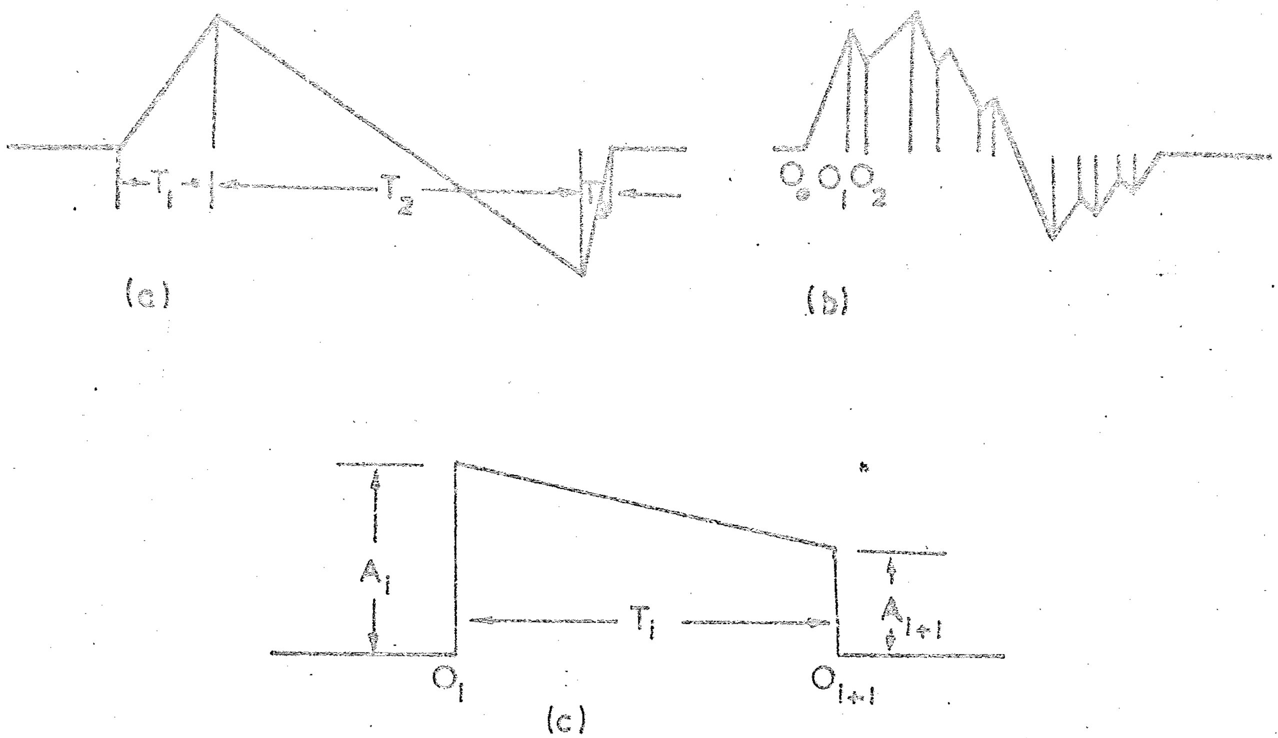


FIGURE 4

(a) A SONIC BOOM TRACE WHICH IS A MODIFIED FORM OF THE 'N' WAVE, WHERE THE SHARP SHOCKS AT THE FRONT AND THE END ARE REPLACED BY FINITE DURATIONS OF RISE AND FALL TIMES.

(b) A FURTHER MODIFIED FORM. A STRAIGHT LINE REPRESENTATION IS ASSUMED.

(c) A MAGNIFIED PART OF (b), WHICH IS TAKEN AS A BASIC UNIT TO FIND THE RESPONSE.

3.3 Response to a General, Sonic Boom Type Transient

In the last section, the response of the boom due to an 'N' wave type excitation, in the (o,o) mode was found. Often, the observed sonic boom traces are seen to possess finite rise and fall times (see Figure 4(a)). A further modification of the detail is seen in Figure 4(b). Under these conditions it might be justifiable to approximate these wave forms by straight lines. (If not, one could combine straight lines and parabolas and so on ..!). Thus a typical segment can be represented by (c) in Figure 4. The initial ordinate is A_i and the final one is A_{i+1}. The time elapsed is T_i. Time will be measured from the instant represented by O_i. Finally, the response of all the individual elements shall be added, making the necessary corrections for the shifting of origin. Therefore, the excitation can be represented by

$$I(t) = \left[A_i - \frac{(A_i - A_{i+1})}{T_i} t\right] + H(t) \left[1 - H(t-T_i)\right]. \tag{3.3.1}$$

Identical procedure, as in the previous section, leads to the following generalization of the result (3.2.26).

$$\frac{\bar{R}^{\bar{n},1}_{o,o}(t,z)}{2\sigma_{o,o}} = (-1)^{\bar{n}} \left\{A_i - \frac{(A_i - A_{i+1})}{T_i}\left(t - \frac{\bar{z}^{\bar{n},1}}{c}\right)\right\} \times \left[-K_{\bar{n}}\left(\frac{t - \frac{\bar{z}^{\bar{n},1}}{c}}{\eta_{o,o}}\right) + K_{\bar{n}}\left(\frac{t - T_i - \bar{z}^{\bar{n},1}/c}{\eta_{o,o}}\right)\right]$$

$$+ (-1)^{\bar{n}} (2\bar{n}+1) \frac{(A_i - A_{i+1})\eta_{o,o}}{T_i} \times \left[-L_{\bar{n}}\left(\frac{t - \bar{z}^{\bar{n},1}/c}{\eta_{o,o}}\right) + L_{\bar{n}}\left(\frac{t - T_i - \bar{z}^{\bar{n},1}/c}{\eta_{o,o}}\right)\right], \tag{3.3.2}$$

and a similar result holds good for the superscript $(\bar{n},2)$.

3.4 Higher Order Modes ($\alpha_{m,n}\eta_{m,n} < 1$)

Consider, once again, the N wave type excitation given by equation (3.2.2). The component of the response,

The component of the response, $\bar{R}^{o,1}_{m,n}(t,z)$ to this excitation for $\frac{z}{c} < t < T + \frac{z}{c}$, will be given by

$$\left[\frac{\bar{R}^{o,1}_{m,n}(t,z)}{2\sigma_{m,n}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}}\right] = \int_{z/c}^{t} \left[1 - \frac{2(t-\tau)}{T}\right]\bar{S}^{o,1}_{m,n}(t,z)\ d\tau. \tag{3.4.1a}$$

$$\equiv \bar{r}^{o,1}_{m,n}(t,z) \tag{3.4.1b}$$

say.

Expression (2.4.13) can be used if $\alpha_{m,n}\eta_{m,n} < 1$. In that case, consider first the integral

$$F_1 \equiv \int_{z/c}^{t} \left[1 - \frac{2(t-\tau)}{T}\right] \frac{J_1(\alpha\sqrt{\tau^2 - \frac{z^2}{c^2}})}{\sqrt{\tau^2 - \frac{z^2}{c^2}}} \, d\tau \quad . \qquad (3.4.2)$$

This integral can be expanded in two ways. The first way is to integrate the term

$$\left[1 - \frac{2(t-\tau)}{T}\right]$$

and differentiate the other term. A readjustment is made after every partial integration, on much similar lines as those in previous sections.

This leads to

$$F_1 = \frac{1}{2} \alpha_{m,n} t(1 - \frac{t}{T}) \left(\frac{2 J_1(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}})}{\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}}}\right)$$

$$+ \frac{1}{24} (\alpha_{m,n} t)^3 (1 - \frac{5}{4}\frac{t}{T}) \left(\frac{8 J_2(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}})}{\alpha_{m,n}^2 (t^2 - \frac{z^2}{c^2})}\right)$$

$$+ \frac{1}{720} [\alpha_{m,n} t]^5 (1 - \frac{43t}{24T}) \left(\frac{48 J_3(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}})}{\alpha^3 (t^2 - \frac{z^2}{c^2})^{3/2}}\right) + \cdots\cdots$$

$$- \frac{1}{2} (\frac{\alpha z}{c}) (1 - \frac{2t}{T} + \frac{z}{cT}) - \frac{1}{24} (\frac{\alpha z}{c})^3 (1 - \frac{2t}{T} + \frac{3}{4}\frac{z}{cT})$$

$$- \frac{1}{720} (\frac{\alpha z}{c})^5 (1 - \frac{2t}{T} + \frac{5}{24}\frac{z}{cT}) + \cdots \qquad (3.4.3)$$

It is worth noting that F_1 vanishes when $t = z/c$. This form is felt to be relevant when both $\alpha\eta < 1$ and $\frac{\alpha z}{c} < 1$. The steady state part of F_1 has the most significant term

$$- \frac{1}{2} \left[\frac{\alpha_{m,n} z}{c}\right] (1 - \frac{2t}{T} + \frac{z}{cT}) \; .$$

When this is substituted in the equation (3.4.1b) and the equation (2.4.13) is made use of, it is seen that the contribution to

$$\bar{r}^{o,1}_{m,n} \quad (t,z) \quad ,$$

due to the term containing

$$J_1, \quad \text{is} \quad -\frac{1}{2}\left(\frac{\alpha_{m,n} z}{c}\right)\left(\frac{\alpha_{m,n} z}{c} + \alpha_{m,n}\eta_{m,n}\right)\left(1 - \frac{2t}{T} + \frac{z}{cT}\right) \quad .$$

In view of the assumed smallness of $\frac{\alpha_{m,n} z}{c}$ and $\alpha_{m,n}\eta_{m,n}$ it can be concluded that this contribution is of a smaller magnitude. The contribution due to the other terms is even smaller.

Consider now the second way of expanding the integral F_1, given by the equation (3.4.2). This consists of integrating the term containing the Bessel function and applying the rule of integration by parts, making use of this integration. A repetition of this procedure leads to the following result:

$$F_1 = -\left\{\frac{1}{\alpha_{m,n}}\left(\frac{1}{\tau}\left[1 - \frac{2t}{T}\right] + \frac{2}{T}\right) J_o\left(\alpha\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)\right.$$

$$+ \frac{1}{\alpha^2_{m,n}} \times \frac{1}{\tau^3}\left[1 - \frac{2t}{T}\right]\sqrt{\tau^2 - \frac{z^2}{c^2}}\; J_1\left(\alpha\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)$$

$$\left.+ \frac{1}{\alpha^3_{m,n}} \times \frac{3}{\tau^5}\left[1 - \frac{2t}{T}\right]\left(\tau^2 - \frac{z^2}{c^2}\right) J_2\left(\alpha\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)\right\}^t_{z/c} \quad ,$$

$$= \frac{1}{\left(\alpha_{m,n}\frac{z}{c}\right)}\left(1 - \frac{2\left(t - \frac{z}{c}\right)}{T}\right)$$

$$- \left[\frac{1}{\alpha_{m,n} t} J_o\left(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}}\right) + \frac{1}{\alpha^2_{m,n} t^2}\left[1 - \frac{2t}{T}\right]\frac{\sqrt{t^2 - \frac{z^2}{c^2}}}{t}\right.$$

$$\left.\times J_1\left(\alpha\sqrt{t^2 - \frac{z^2}{c^2}}\right) + \frac{3}{\alpha^3_{m,n} t^3}\left[1 - \frac{2t}{T}\right]\frac{t^2 - \frac{z}{c}}{t} J_2\left(\alpha\sqrt{t^2 - \frac{z^2}{c^2}}\right) + \dots\right]$$

(3.4.4)

The expression has a value of zero when $t = z/c$. The steady state value is given by

$$\frac{1}{(\alpha_{m,n}\ z/c)} \quad (1 - \frac{2}{T}\ (t - \frac{z}{c})).$$

The contribution to $\bar{r}^{o,1}_{m,n}\ (t,z)$, due to this integral is

$$-(1 - \frac{2}{T}\ (t - \frac{z}{c}))\ (1 + \frac{\eta_{m,n}c}{z}).$$

Now

$$\frac{\eta_{m,n}c}{z} = \frac{\alpha_{m,n}\eta_{m,n}}{(\alpha_{m,n}z/c)} \quad . \qquad (3.4.5)$$

This form of the expansion of the integral F_1 is suitable when $\frac{\alpha_{m,n}z}{c} > 1$. Further it would appear from (3.4.5) that $\frac{\eta_{m,n}c}{z}$ is a small quantity if $\alpha_{m,n}\eta_{m,n} < 1$ and $\frac{\alpha_{m,n}z}{c} > 1$. Therefore, the contribution to $\bar{r}^{o,1}_{m,n}(t,z)$, due to the term

$$-H(t - \frac{z}{c})\ \frac{\alpha^2_{m,n}\eta_{m,n}}{2} \quad B_1(t,\frac{z}{c})\ (\frac{z}{c\eta_{m,n}} + 1) \quad ,$$

is very nearly given by

$$\left[1 - \frac{2(t - \frac{z}{c})}{T} \right] .$$

It is interesting to note that this contribution, thus, cancels that due to the term

$$H(t - \frac{z}{c})\ \frac{e^{\frac{-(t-z/c)}{\eta_{m,n}}}}{\eta_{m,n}} .$$

From the form of the equation (2.4.13), it would be apparent that under these conditions

$$(\alpha_{m,n}\eta_{m,n} < 1\ ,\ \frac{\alpha_{m,n}z}{c} > 1) \quad .$$

The contribution to $\bar{R}^{o,1}_{m,n}(t,z)$ due to terms such as $H(t - \frac{z}{c})\ \frac{\alpha^2_{m,n}\eta_{m,n}}{2}$ $\left(\frac{e^{\frac{-(t - \frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}}\ (\frac{t}{\eta_{m,n}} + 1)\right)$ might assume importance. Let us denote this particular term by the symbol N_{exp}.

3.5 Contribution due to 'N_{exp}'

Let the contribution to $\bar{r}^{o,1}_{m,n}(t,z)$ due to $N_{exp} \equiv H(t - \frac{z}{c})\frac{\alpha^2_{m,n}\eta^2_{m,n}}{2}$

$$\times \frac{e^{\frac{-(t - z/c)}{\eta_{m,n}}}}{\eta_{m,n}} \left(\frac{t}{\eta_{m,n}} + 1\right)$$ be denoted by G.

Therefore

$$\frac{2G}{\alpha^2_{m,n}\eta^2_{m,n}} = \left[1 - \frac{2t}{T}\right] \int_{z/c}^{t} \frac{e^{\frac{-(\tau - \frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}} \left(\frac{\tau}{\eta_{m,n}} + 1\right) dt$$

$$+ \frac{2\eta_{m,n}}{T} \int_{z/c}^{t} \frac{e^{\frac{-(\tau - \frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}} \left(\frac{\tau^2}{\eta^2_{m,n}} + \frac{\tau}{\eta_{m,n}}\right) dt \; . \qquad (3.5.1)$$

Let $\theta \equiv \frac{\tau - \frac{z}{c}}{\eta_{m,n}}$. $\qquad d\theta = \frac{d\tau}{\eta_{m,n}}$ etc...

Therefore

$$\frac{2G}{\alpha^2_{m,n}\eta^2_{m,n}} = \left[1 - \frac{2t}{T}\right] \int_{0}^{\frac{t - \frac{z}{c}}{\eta_{m,n}}} e^{-\theta}\left(\theta + \frac{z}{c\eta_{m,n}} + 1\right) d\theta$$

$$+ \frac{2\eta_{m,n}}{T} \int_{0}^{\frac{t - \frac{z}{c}}{\eta_{m,n}}} e^{-\theta} \left(\theta^2 + \theta\left[1 + \frac{2z}{c\eta_{m,n}}\right] + \left(\frac{z^2}{c^2\eta^2} + \frac{z}{c\eta}\right) d\theta\right. ,$$

$$= \left[2 + \frac{z}{c\eta_{m,n}} - e^{\frac{-(t - \frac{z}{c})}{\eta_{m,n}}} \left(\frac{t}{\eta_{m,n}} + 2\right)\right] \left(1 - \frac{2t}{T}\right)$$

(continued over..)

$$+\frac{2\eta_{m,n}}{T}\left[\left(3+\frac{3z}{c\eta_{m,n}}+\frac{z^2}{c^2\eta_{m,n}^2}\right)-e^{\frac{-(t-\frac{z}{c})}{\eta_{m,n}}}\left(3+\frac{t}{\eta_{m,n}}+3\frac{t^2}{\eta_{m,n}^2}\right)\right]$$

(3.5.2)

As $(t - z/c)/\eta_{m,n} >> 1$, the expression

$$\left[2+\frac{z}{c\eta_{m,n}}\right]\left[1-\frac{2t}{T}\right]+\frac{2\eta_{m,n}}{T}\left(3+\frac{3z}{c\eta_{m,n}}+\frac{z^2}{c^2\eta_{m,n}^2}\right).$$ Therefore

the contribution, after $(t - \frac{z}{c}) >> \eta_{m,n}$, is

$$G = \left(1-\frac{2t}{T}\right)\left(\alpha_{m,n}^2\eta_{m,n}^2+\frac{\alpha_{m,n}\eta_{m,n}}{2}\left(\frac{\alpha_{m,n}z}{c}\right)\right.$$

$$\left.+\frac{\eta_{m,n}}{T}\left(3\alpha_{m,n}2\eta_{m,n}^2+\frac{3}{2}\left(\alpha_{m,n}\eta_{m,n}\right)\frac{\alpha_{m,n}z}{c}+\frac{1}{2}\frac{\alpha_{m,n}^2z^2}{c^2}\right)\right)$$

(3.5.3)

It has been assumed that $\alpha_{m,n}\eta_{m,n} < 1$. Therefore, it appears that if $\frac{\eta_{m,n}}{T} < 1$, (and this is so for most practical cases encountered in the study of the response to sonic booms since the period of the boom, T, is, in fact, often quite large) G will be very small as long as $\frac{\alpha_{m,n}z}{c}$ is also less than 1.

When $\frac{\alpha_{m,n}z}{c} > 1$, G may not be small. We now arrive at composite parameters such as

$$\frac{\alpha_{m,n}^2\eta_{m,n}z}{c} \quad \text{and} \quad \frac{\alpha_{m,n}^2\eta_{m,n}z^2}{Tc^2}.$$

If all these parameters happen to be much less than unity, G will be small.

A generalization, of the result given by equation (2.3.8), will soon be made. In that generalization, z is replaced by the distance of the image source from the point under observation, (equal to either $(2\bar{n}d + z)$ or $([2\bar{n}+2]d-z)$). As time increases, more and more terms come into operation. Thus the equivalent of z becomes larger and larger. It thus seems possible that after sufficient lapse of time, terms containing large enough values of z, would have come into operation. Thus the magnitude of G would far exceed unity. When we consider $R_{m,n}(t,z)$, the behaviour would be further modified due to the presence of its coefficients :

$$\sigma_{m,n} \quad , \quad \cos \frac{m\pi x}{a} \quad , \quad \cos \frac{n\pi y}{b} \quad .$$

The last two would only cause spatial variation. The first one, the modal distribution parameter, would, in a sense, restrict the error caused if the effect of the term N_{exp} is neglected.

It can be concluded, that for modes with small enough $\sigma_{m,n}$, in the earlier part of the time history, the contribution to $R_{m,n}(t,z)$ due to N_{exp} can be neglected if $\alpha_{m,n}\eta_{m,n} << 1$. For a later part of the time history the omission of the term would lead to more and more errors. The fact that the terms such as N_{exp} do not arise when $\alpha_{m,n}\eta_{m,n} > 1$ thus has important consequences in applying the theory to practical situations.

3.6 Response to an N-Wave ($\alpha_{m,n}\eta_{m,n} > 1$)

It appears from the form of equation (2.5.6) that, apart from the integral involving the exponential term :

$$\frac{e^{\frac{-(t - \frac{z}{c})}{\eta_{m,n}}}}{\eta_{m,n}}$$

the integrals to be evaluated to compute $\bar{r}^{o,1}_{m,n}(t,z)$ will contain Bessel Functions. Consider the following integral, denoted by the symbol F_2:

$$F_2 = \int_{z/c}^{t} \left[1 - \frac{2(t-)}{T}\right] \left[J_{o\ m,n} \sqrt{\tau^2 - \frac{z}{c}}\right] dt \qquad (3.6.1)$$

Once again there are two possible ways of evaluating this integral. When the first bracket is integrated, the results obtained appear to be satisfactory for

$$\frac{\alpha_{m,n} z}{c} < 1 \ ;$$

and when the second bracket is chosen for integration, the results seem satisfactory for

$$\frac{\alpha_{m,n} z}{c} > 1 \quad .$$

The first method leads to

$$\alpha_{m,n}F_2 = \alpha\Big\{\left[\tau\left[1 - \frac{2t}{T}\right] + \frac{\tau^2}{T}\right] J_o\left(\alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)$$

$$+\alpha\left[\frac{\tau^3}{3}\left[1 - \frac{2t}{T}\right] + \frac{\tau^4}{4T}\right] J_1\left(\alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)$$

$$+ \tau^2 \left[\frac{\tau^5}{5}\left(1 - \frac{2t}{T}\right) + \frac{\tau^6}{24T}\right] \left.\frac{J_2\left(\alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)}{\left(\tau^2 - \frac{z^2}{c^2}\right)}\right|_{z/c}^{t} \quad . \tag{3.6.2}$$

The identities used to obtain this result are

$$\frac{d}{d\tau} J_o\, \alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}} = \frac{\alpha_{m,n}\tau\, J_1\left(\alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)}{\sqrt{\tau^2 - \frac{z^2}{c^2}}} \quad , \tag{3.6.3}$$

$$\frac{d}{d\tau}\ \frac{J_1\left(\alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)}{\sqrt{\tau^2 - \frac{z^2}{c^2}}} = \frac{\alpha_{m,n}\tau J_2\left(\alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)}{\left(\tau^2 - \frac{z^2}{c^2}\right)} \quad , \tag{3.6.4}$$

$$\frac{d}{d\tau}\ \frac{J_2\left(\alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)}{\sqrt{\tau^2 - \frac{z^2}{c^2}}} = \frac{\alpha_{m,n}\tau\, J_3\left(\alpha_{m,n}\sqrt{\tau^2 - \frac{z^2}{c^2}}\right)}{\left(\tau^2 - \frac{z^2}{c^2}\right)^{3/2}} \quad , \tag{3.6.5}$$

etc...

The substitution of the limits in the equation (3.6.2) leads to

$$\alpha_{m,n}F_2 = \Big\{\alpha_{m,n}\, t\left(1 - \frac{t}{T}\right) J_o\left(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}}\right)$$

$$+ \frac{1}{6}\left(\alpha_{m,n}t\right)^3 \left[1 - \frac{5}{4}\frac{t}{T}\right] \cdot \left[\frac{2\, J_1\left(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}}\right)}{\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}}}\right]$$

$$+ \frac{1}{40} (\alpha_{m,n} t)^5 \left[1 - \frac{43}{24}\frac{t}{T}\right] \left(\frac{8 J_2(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}})}{\alpha_{m,n}^2 (t^2 - \frac{z^2}{c^2})}\right) \}$$

$$- \{\frac{\alpha_{m,n} z}{c} \left[1 - \frac{2t}{T} + \frac{z}{cT}\right] + \frac{1}{6} (\frac{\alpha_{m,n} z}{c})^3 \left[1 - \frac{2t}{T} + \frac{3}{4} \frac{z}{cT}\right]$$

$$+ \frac{1}{40} (\frac{\alpha_{m,n} z}{c})^5 \left[1 - \frac{2t}{T} + \frac{5}{24} \frac{z}{cT}\right] + \ldots \} \quad (3.6.6)$$

The main contribution to $\bar{r}^{o,1}_{m,n}(t,z)$, due to this part, is

$$= \frac{-1}{\alpha_{m,n}\eta_{m,n}} \{\frac{\alpha_{m,n} z}{c} \left[1 - \frac{2t}{T} + \frac{z}{cT}\right] - \alpha_{m,n} t \left(1 - \frac{t}{T}\right) J_o(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}})\} ,$$

provided $\frac{\alpha_{m,n} z}{c} < 1$. It can be seen from this expression that if $\alpha_{m,n}\eta_{m,n}$ and $\frac{\alpha_{m,n} z}{c}$ are much less than unity, the contribution due to the term containing the Bessel function is negligible.

The other method of integration of equation (3.6.1) leads to the following result :

$$\alpha_{m,n} F_2 = \frac{\sqrt{t^2 - \frac{z^2}{c^2}}}{t} J_1 (\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}})$$

$$+ \frac{1}{\alpha_{m,n} t} \left[t - \frac{2t}{T}\right] \left(\frac{t^2 - \frac{z^2}{c^2}}{t^2}\right) J_2(\alpha_{m,n}\sqrt{t^2 - \frac{z^2}{c^2}})$$

$$+ \frac{3}{\alpha_{m,n}^2 t^2} \left[1 - \frac{2t}{T}\right] \left(\frac{(t^2 - \frac{z^2}{c^2})^{3/2}}{t^3} J_3 \left|\alpha\sqrt{t^2 - \frac{z^2}{c^2}}\right|\right)$$

$$+ \ldots \quad (3.6.7)$$

The terms form a decreasing sequence when $\frac{\alpha_{m,n} z}{c} > 1$. (Thus the minimum value of $\alpha_{m,n} t$ is greater than unity. By means of a comparison with a geometric series, the convergence of the above expression can be established in the above equation).

It is apparent that $\alpha_{m,n} F_2$ has no steady component. Also, the contribution, due to the term containing J_o in

$$\frac{\alpha_{m,n} F_2}{\alpha_{m,n} \eta_{m,n}}$$

is small and becomes soon very negligible when $t >> \frac{z}{c} + \eta_{m,n}$.

Other integrals, involving terms containing J_1, J_2 etc., can be evaluated by an identical procedure and the same conclusions would be obtained.

3.7 Approximate Response to an N Wave

The equation (2.3.8) can be interpreted as follows. The first term in the square brackets corresponds to the axial-mode type of response. The second term corresponds to a deviation from it. The variation critically depends upon the non-dimensional parameters $(\alpha_{m,n} \eta_{m,n})$ and $(\frac{\alpha_{m,n} z}{c})$. (When the response to an N wave is considered a further parameter $(\frac{\eta_{m,n}}{T})$, enters the discussion). From this point of view the following possibilities arise.

i) The contribution due to the correction term might be negligible. In this case the axial-type term predominates and the final contribution can be considered to be approximately equal to the contribution calculated by using the same formula as for the axial modes with the substitution of $\eta_{m,n}$ for $\eta_{o,o}$ and $\sigma_{m,n} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$ for $\sigma_{o,o}$.

This happens in the very near field of the window, i.e. for $\frac{\alpha_{m,n} z}{c} << 1$, (see equations (3.6.6) and (3.4.3) followed by the discussions).

This also happens when $\alpha_{m,n} \eta_{m,n} >> 1$. This condition is generally satisfied by modes with smaller values of m and n. This observation may be associated with another observation in the previous report (1). It was found there that the parameter $k^*_{m,n}$ would become imaginary below a certain cut-off frequency, $(= \frac{ck^*_{m,n}}{2\pi})$, and that this would alter the characteristics of transmission. Thus $\alpha_{m,n} \eta_{m,n} >> 1$ appears to be a modified condition which states that the substantial part of the spectrum corresponds to those frequencies which are above cut-off.

ii) Another possibility is that the contribution due to the second term might almost exactly cancel the contribution due to the first term. This happens when $\alpha_{m,n} \eta_{m,n} >> 1$ and $\frac{\alpha_{m,n} z}{c} >> 1$, and the time elapsed is not very large (to be specific $\{\frac{\alpha_{m,n}{}^2 \eta_{m,n} z}{c}\} < 1$ etc.).

iii) Another possibility is that the contribution due to the second term would be of a significantly higher magnitude and needs separate computation.

This happens when $\alpha_{m,n}\eta_{m,n} << 1$ and long enough time has elapsed.

So far only $\bar{r}^{o,1}_{m,n}(t,z)$ has been discussed. The case of $\bar{r}^{o,2}_{m,n}(t,z)$ remains identical if for z, $(2d-z)$ is substituted. The integrals involved can be evaluated for the other terms by an identical procedure and very similar results are obtained. The conclusion can be summarised as follows :

$$\bar{r}^{\bar{n},1}_{m,n}(t,\bar{z}^{\bar{n},1}) \equiv \frac{\bar{R}^{\bar{n},1}_{m,n}(t,\bar{z}^{\bar{n},1})}{2\sigma_{m,n}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}} , \qquad (3.7.1)$$

$$\bar{r}^{\bar{n},1}_{m,n}(t,\bar{z}^{\bar{n},1}) \approx \bar{r}^{\bar{n},1}_{o,o}\left[t,\bar{z}^{\bar{n},1}\right]_{\eta_{o,o}=\eta_{m,n}} ; \qquad (3.7.2)$$

if either $(\alpha_{m,n}\eta_{m,n} > 1)$ or $(\alpha_{m,n}\eta_{m,n} < 1$ and $\frac{\alpha_{m,n}\bar{z}^{\bar{n},1}}{c} << 1)$;

$$\bar{r}^{\bar{n},1}_{m,n}(t,\bar{z}^{\bar{n},1}) \approx 0 ; \qquad (3.7.3)$$

if $(\alpha_{m,n}\eta_{m,n} < 1)$, and for the earlier part of the time history. In these expressions

$$\bar{z}^{\bar{n},1} = z + 2\bar{n}d , \qquad (3.7.4)$$

$$\bar{z}^{\bar{n},2} = -z + (2\bar{n} + 2)\, d. \qquad (3.7.5)$$

Substitution of the superscript $(\bar{n},2)$ instead of $(\bar{n},1)$ in equations (3.7.1), (3.7.2) and (3.7.3) would lead to equations which correspond to $\bar{z}^{\bar{n},2}$.

These equations can be used to compute the response of the room. This response is quite accurate for the earlier part of the pressure time history. For the later part, in the limit, the Helmholtz-resonator behaviour to be described in the subsequent sections, will be observed.

3.8 The Response to an Incoming Wave, using Helmholtz Resonator Analogy

In Section 3.3, a method of representing incoming sonic boom signals was described. First of all, the response to a 'basic' incoming wave, represented by Figure 4(c) will be found. To a good, and usually sufficient, degree of accuracy, an arbitrary input, such as in Figure 4(b) can be built up from such basic units. (The conventional 'N' wave is a particular case of the basic unit waveform when $A_i = + A$, $A_{i+1} = -A$ and $T_i = T$).

Now if the origin be placed at O_i, the waveform can be expressed as

$$N(t) = A_i \left[1 + \beta\Omega t\right] , \quad (o < t < T_i) \tag{3.8.1}$$

and

$$N(t) = o , \quad (o > t \text{ or } t > T_i) ; \tag{3.8.2}$$

where

$$\beta = \frac{A_{i+1} - A_i}{A_i \ \Omega T_i} . \tag{3.8.3}$$

β is evidently a dimensionless quantity.

From equations (2.6.8), (3.8.1) and (3.8.2), it is seen that

$$\xi(t) = o \qquad t < o \tag{3.8.4}$$

$$= \frac{2A}{L\Omega} \int_o^t e^{-\mu(t-\tau)} \sin \Omega(t-\tau) \left[1 + \beta\Omega\tau\right] d\tau , \quad (o < t < T_i) , \tag{3.8.5}$$

$$= \frac{2A}{L} \int_o^{T_i} e^{-\mu(t-\tau)} \sin \Omega(t-\tau) \left[1 + \beta\Omega\tau\right] d\tau , \ (t > T_i). \tag{3.8.6}$$

Further, let

$$\Omega t = \theta , \tag{3.8.7}$$

$$\Omega\tau = \psi , \tag{3.8.8}$$

and

$$\mu' = \mu/\Omega . \tag{3.8.9}$$

Define a quantity θ' with the following properties:

$$\theta' = o , \quad (\text{if } t < o) , \tag{3.8.10}$$

$$= \theta , \quad (\text{if } o < t < T_i) , \tag{3.8.11}$$

$$= \Omega T_i \equiv \theta_i , \ (\text{if } t > T_i) . \tag{3.8.12}$$

With the help of these definitions, the various expressions for $\xi(t)$ can be represented as

$$\xi(t) = \frac{2A_i}{L\Omega^2} \int_o^{\theta'} e^{-\mu'(\theta-\psi)} \sin(\theta-\psi) \times \left[1 + \beta\psi\right] d\psi . \tag{3.8.13}$$

The integral is evaluated by the method of integration by parts. Thus

$$\xi(t) = \frac{2A_i}{L\Omega^2} \frac{e^{-\mu'(\theta-\theta')}}{1+\mu'^2} \{[1 + \beta(\theta' - \frac{2\mu'}{1+\mu'^2})] \cos(\theta-\theta')$$

$$+ [1 + \beta(\theta' + \frac{1-\mu'^2}{\mu'(1+\mu'^2)})] \mu' \sin(\theta-\theta')$$

$$\frac{-e^{-\mu'\theta}}{1+\mu'^2} \left[[1 - \beta\frac{2\mu'}{1+\mu'^2}] \cos\theta + [1 + \beta\frac{1-\mu'^2}{\mu'(1+\mu'^2)} \mu' \sin\theta]\right] .$$

(3.8.14)

Usually, μ', $(= \mu/\Omega)$ is much smaller than unity. Under this condition, with the appropriate substitution for θ', the following expressions for $\xi(t)$ are obtained :

$$\xi(t) = o, \quad (t < o) \tag{3.8.15}$$

$$\xi(t) = \frac{2A_i}{L\Omega^2} \left[(i - e^{-\mu'\theta} \cos\theta) + \beta\theta[1 - e^{-\mu'\theta} \frac{\sin\theta}{\theta}]\right] \quad (o < t < T_i) \tag{3.8.16}$$

and

$$\xi(t) = \frac{2A_i}{L\Omega^2} \Big[e^{-\mu'(\theta-\theta_i)} \{(1 + \beta\theta_i) \cos(\theta-\theta_i) + \beta \sin(\theta-\theta_i)\} - e^{\mu'\theta} \{\cos\theta + \beta \sin\theta\}\Big] , \quad (t > T_i) . \tag{3.8.17}$$

The pressure inside the room is given by $\dfrac{\xi(t)}{C_p}$. Now the coefficient

$$\frac{2A_i}{L\Omega^2} = 2A_i \quad C_p \quad \{ \frac{1}{C_p \; L\Omega^2} \}$$

$$= 2A_i \quad C_p \quad \{ \frac{1}{1 - \mu^2 C_p L} \}$$

$$= 2A_i \quad C_p \quad \{ \frac{1}{1 - \mu'^2} \} . \tag{3.8.18}$$

Therefore from equations (3.8.16) and (3.8.17) and (3.8.18), the pressure inside the room, p_i, is given by

$$p_i = \frac{2}{1-\mu'^2}\left[A_i\,(1 - e^{-\mu'\theta}\cos\theta) + \beta\theta\left[1 - e^{-\mu'\theta}\,\frac{\sin\theta}{\theta}\right]\right] \quad (o < t < T_i)$$

(3.8.19)

and

$$p_i = \frac{2A_i}{1-\mu'^2}\left[e^{-\mu'(\theta-\theta_i)}\{(1+\beta\theta_i)\cos(\theta-\theta_i) + \beta\sin(\theta-\theta_i)\} - e^{-\mu'\theta}\{\cos\theta + \beta\sin\theta\}\right]. \qquad (3.8.20)$$

μ'^2 is already assumed to be much lesser than unity. Therefore, the coefficient outside the brackets could well be taken to be $2A_i$.

3.9 Numerical Results, using Normal Modes

If the window were to extend throughout the width and the height of the room, only axial modes (designated as (o,o)) will be present, provided the assumption of a uniform velocity over the cross-section is made.

For such a window, in a room of typical dimensions, both the attached inertial mass, and also the resistive component of its radiation impedance, are small for frequencies less than 2C/L, where L is window perimeter. If it is assumed to be zero, and if the response to an incoming, N-wave type, signal is calculated, the results would be as shown in Figure 5.

The reflection pattern is very clearly seen. The points nearer to the window receive the signal earlier and the reflected signal from the end wall later. Reflections from the window involve a phase change of 180° and the backward and forward reflections persist undiminished in the 'free oscillation' period.

The idealized picture would undergo modifications when the inertia of the opening, the reradiation of energy out of the window and the absorption of energy inside the room were considered.

The inertial effects would 'round up' the build up proportions, by imposing a behaviour on the build up which is analogous to the switching transients in electricity. The resistive elements would lead to a diminishing of the amplitudes all along the time history.

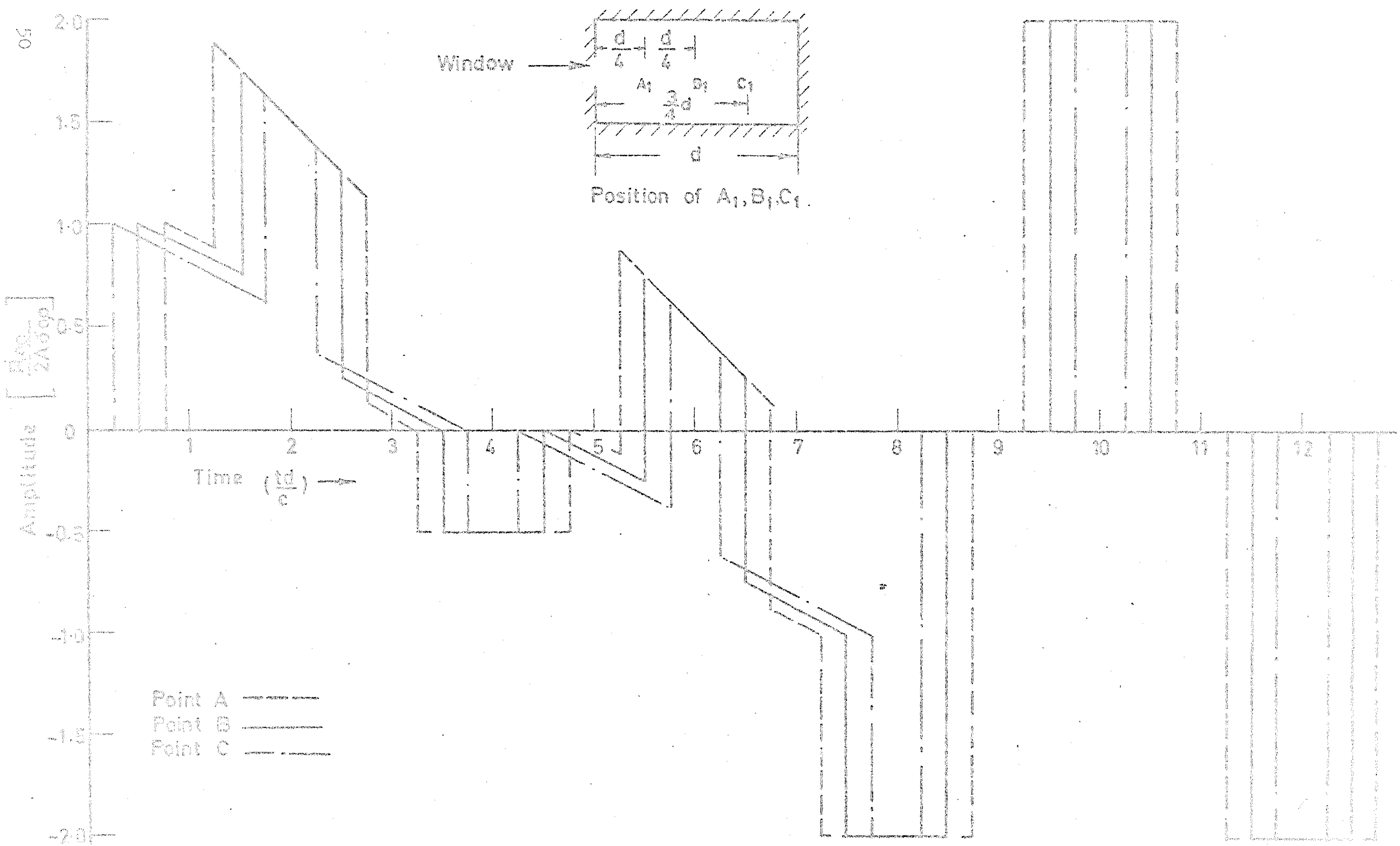


FIGURE 5 IDEALIZED RESPONSE TO AN N WAVE IN THE (0,0) MODE AT THREE POINTS INSIDE THE ROOM FOR $T = 8(d/c)$.

THE RADIATIONAL IMPEDANCE AT THE WINDOW IS ASSUMED TO BE ZERO.

The results obtainable by the use of such a procedure can be illustrated by means of Figures 6 and 7.

It has been seen that other modes, which are not included in this computation, have a very small magnitude at very early time histories. One such case is illustrated in Figure 8. After sufficient time, when all such 'other' modes have appreciable magnitudes, the Helmholtz resonator analogy can be justifiably used. (for the room cited in the example in the next section, the sufficient time would be about 100 m.s.).

3.10 Numerical Results, using Resonator Analogy

Numerical calculation, using the resonator analogy approach, is relatively much simpler.

A specific case is shown in Figure 9. The volume of the room was 2430cft. The window area was 12 sq.ft. The incident boom had a time duration of 200 m.s. Four different values of the parameter μ are used.

3.11 Effect of 'Initial Rise Time'.

For the out-door sonic boom investigations on the subjective response (8) showed the importance of the initial rise time.

Calculations for the very early part of the time-history of the indoor response, for various initial rise times of the N-waves, are presented in Figure 10. These have been computed by using the technique shown in Figure 4.

4.0 CONCLUSIONS

It has been shown that once characteristics of the boom and geometry of the room are known, the characteristics of the transient pressure field inside the room can be predicted.

Two approaches have been found to be relevant. The first one consists of obtaining the steady state response using the normal mode technique (see ref.1) and converting this response into the time domain. The second method is based on the Helmholtz-resonator analogy.

Both methods can be used together to give the complete response to a general transient excitation of the room. Each method has its own advantages and disadvantages.

Experimental results along with a general discussion will be presented in the third and final part of this series of reports.

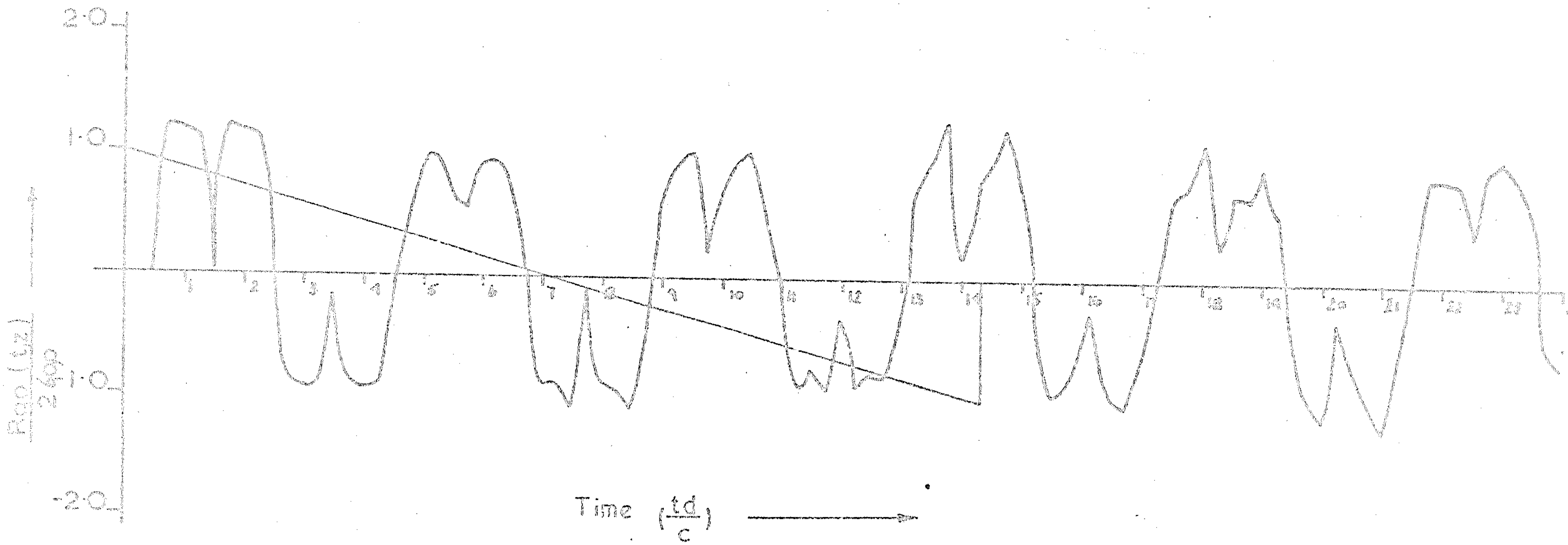


FIGURE 6 THE CONTRIBUTION OF THE (0,0) MODE TO THE RESPONSE AT THE CENTRE OF THE ROOM TO AN 'N' WAVE. THE WINDOW IS 3' x 3'.

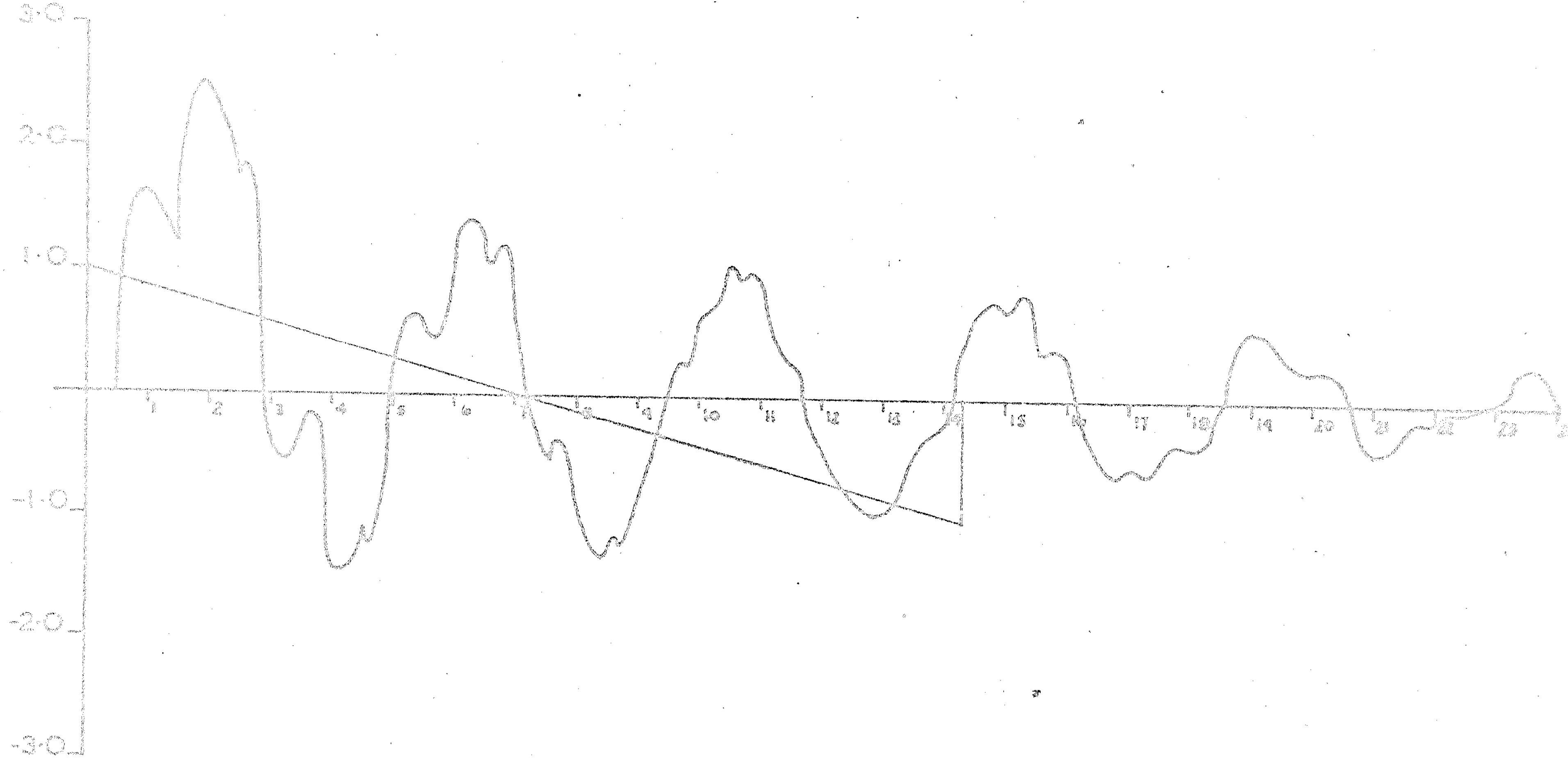


FIGURE 7 THE SUM OF THE CONTRIBUTIONS OF SOME SPECIFIC MODES. THE EARLY TIME HISTORIES ARE EXPECTED TO BE DOMINATED BY SUCH MODES.

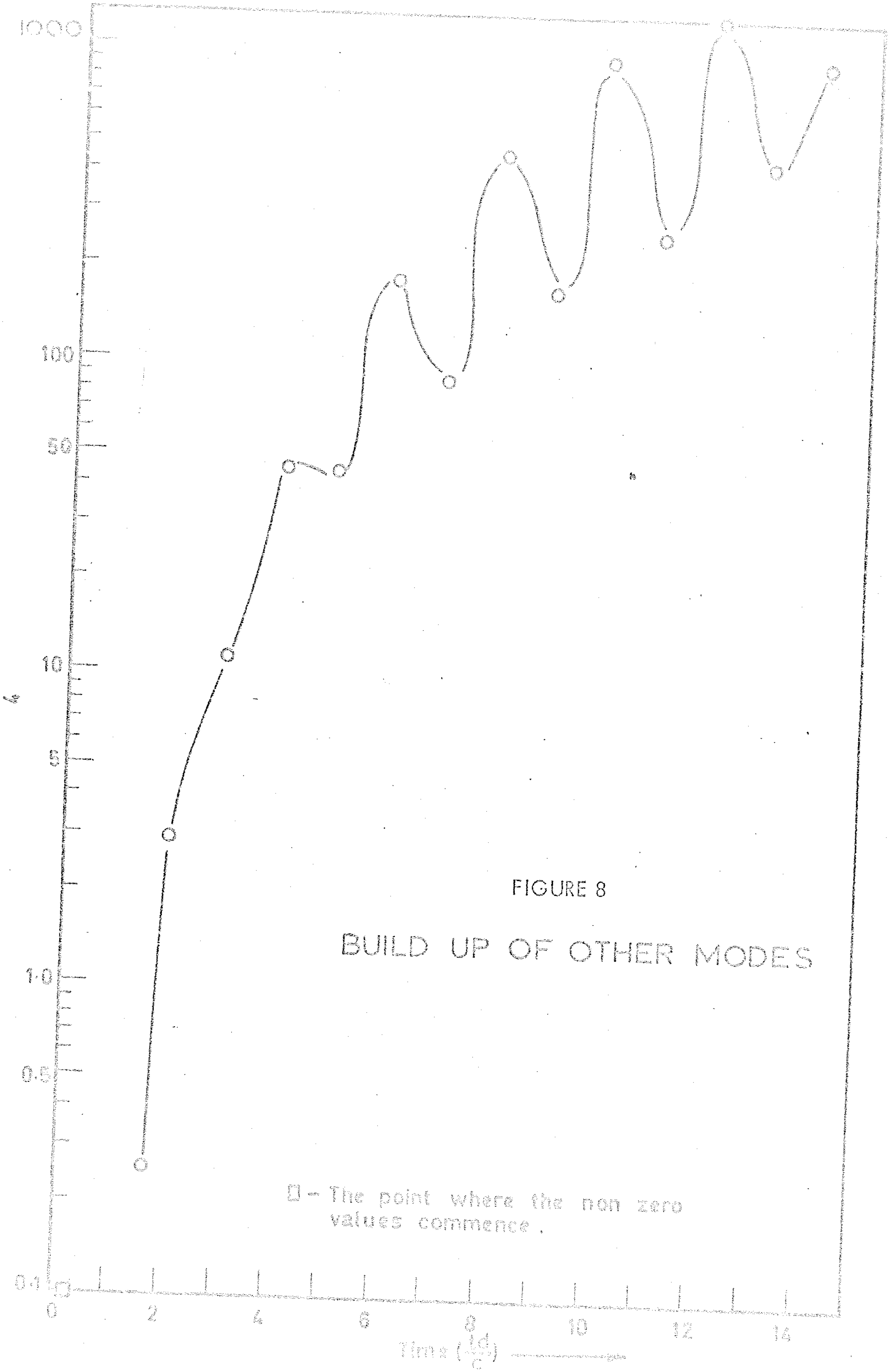


FIGURE 8

BUILD UP OF OTHER MODES

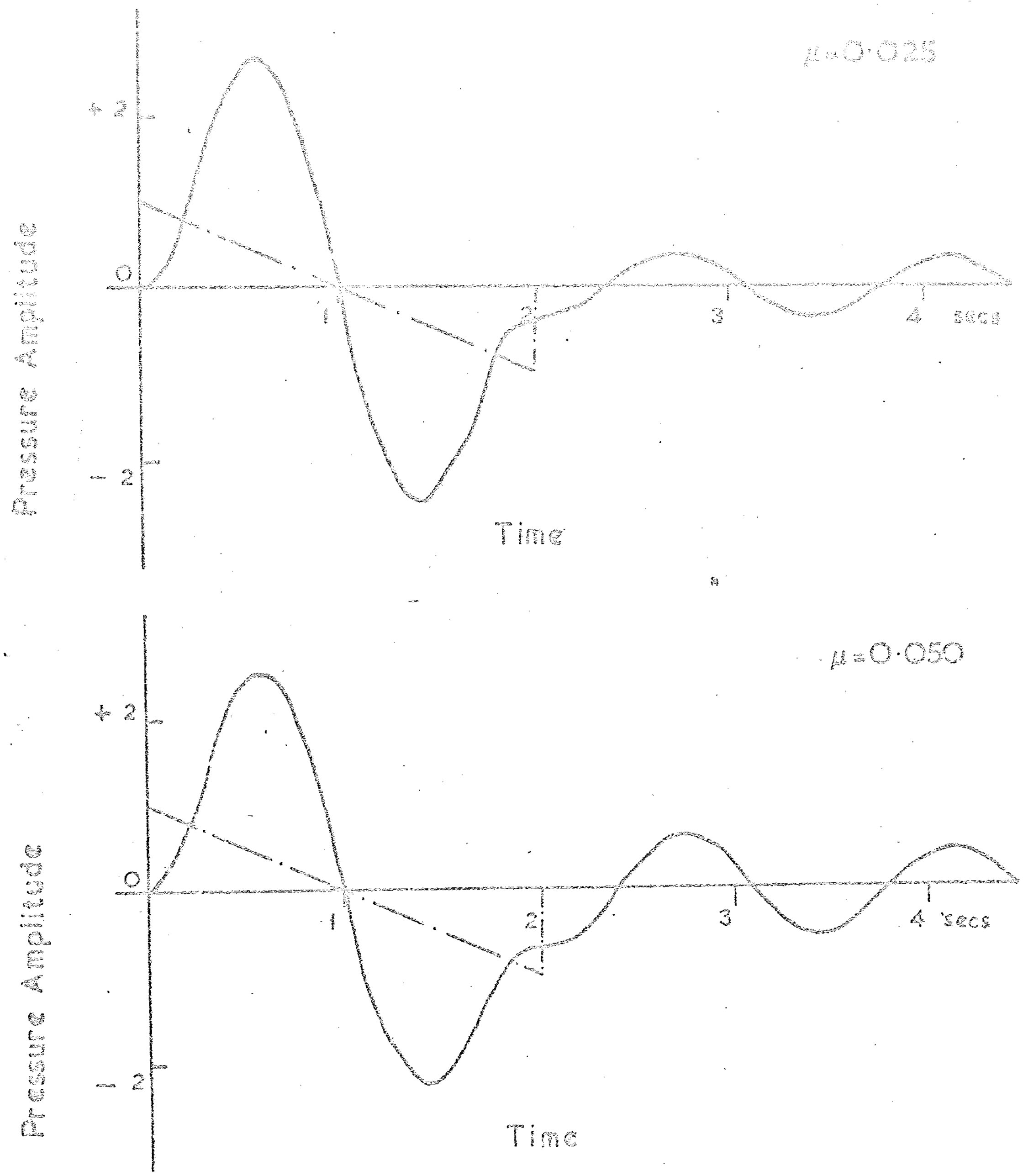


FIGURE 9.1 PRESSURE FIELD INSIDE THE ROOM COMPUTED BY USING THE RESONATOR-TYPE FORMULA. THE ROOM HAS A VOLUME OF 2430 c.f.t. AND A WINDOW AREA OF 12 SQ. FT. DURATION OF THE BOOM IS 200 MSECS. FOUR DIFFERENT VALUES OF THE DAMPING PARAMETER ARE USED

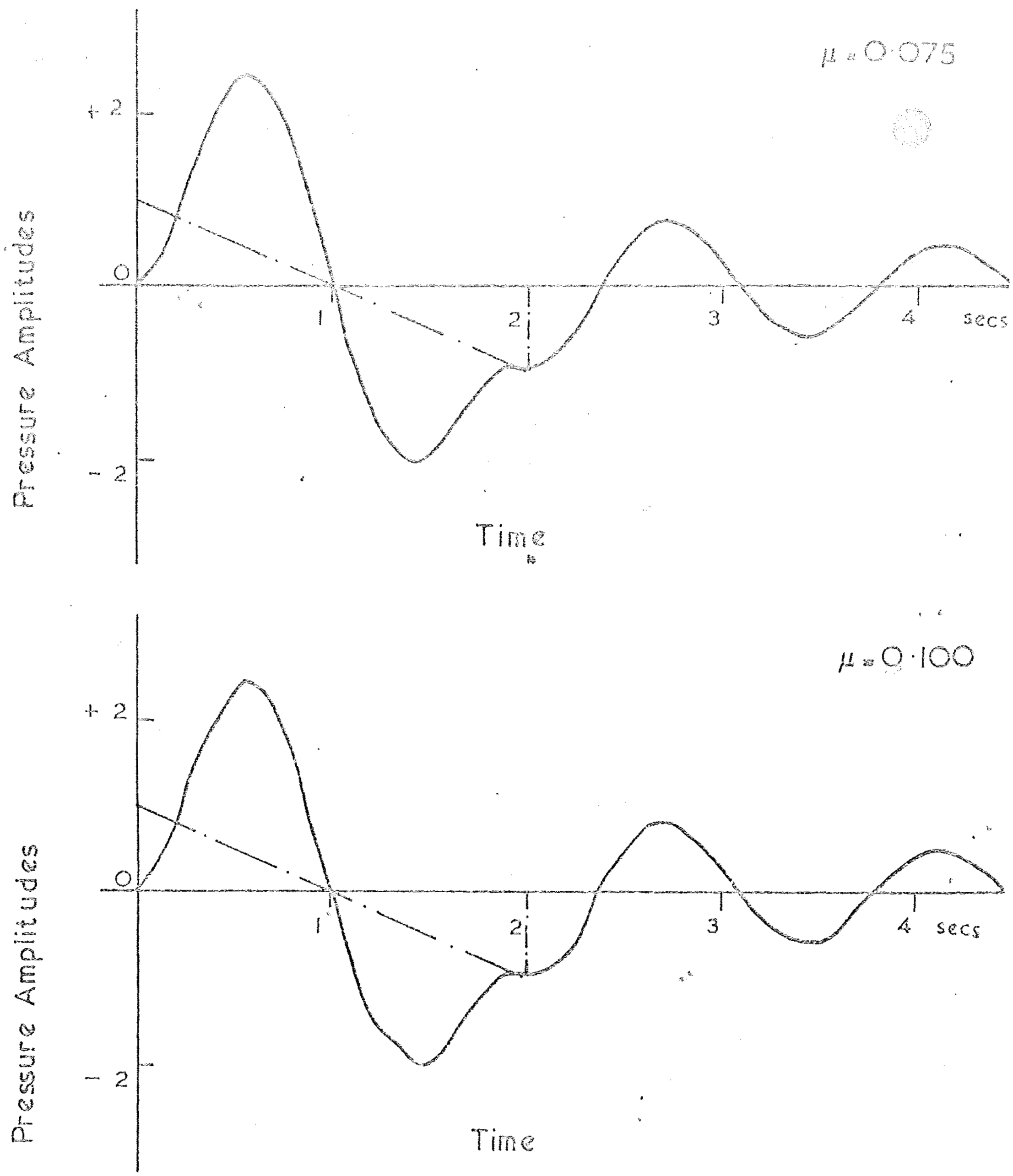
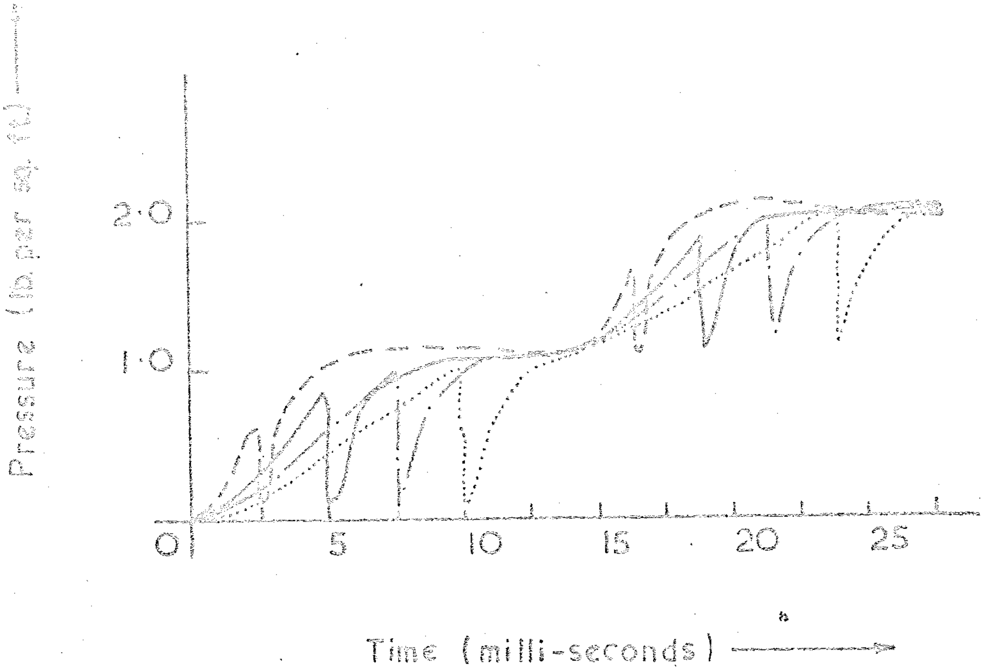


FIGURE 9.2 PRESSURE FIELD INSIDE THE ROOM COMPUTED BY USING THE RESONATOR-TYPE FORMULA. μ IS THE DAMPING COEFFICIENT.



Rise Times

- - - - - -	2 · 5 milliseconds	
————	5 · 0	"
—— · ——	7 · 5	"
....	10 · 0	"

FIGURE 10 EFFECT OF INITIAL RISE TIME ON THE EARLY PART OF THE PRESSURE TIME HISTORY IN THE ROOM.

FOUR NOMINAL 200 M.S. BOOMS WITH PEAK AMPLITUDE 1 P.S.F. AND VARIOUS RISE TIMES ARE SHOWN

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